

### Abstract

The Thurston compactification of Teichmüller spaces has been generalized to many different representation spaces by J. Morgan, P. Shalen, M. Bestvina, F. Paulin, A. Parreau and others. In the simplest case of representations of fundamental groups of closed hyperbolic surfaces in  $PSL(2, \mathbb{R})$ , we prove that this compactification is very degenerated: the nice behaviour of the Thurston compactification of the Teichmüller space contrasts with wild phenomena happening on the boundary of the other connected components of these representation spaces. We prove that it is more natural to consider a refinement of this compactification, which remembers the orientation of the hyperbolic plane. The ideal points of this compactification are *fat*  $\mathbb{R}$ -trees, *i.e.*,  $\mathbb{R}$ -trees equipped with a planar structure.

# Connected components of the compactification of representation spaces of surface groups

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# 1 Introduction

Let  $\Gamma$  be a group with a given finite generating set  $S$ . In all this text, we consider the space  $R_\Gamma(n) = \text{Hom}(\Gamma, \text{Isom}^+(\mathbb{H}^n))$  of actions of  $\Gamma$  on the real hyperbolic space of dimension  $n$  by isometries preserving the orientation. The set  $R_\Gamma(n)$  naturally embeds in  $(\text{Isom}^+(\mathbb{H}^n))^S$ , giving it a Hausdorff, locally compact topology. The Lie groups  $\text{Isom}^+(\mathbb{H}^n)$  and  $\text{Isom}(\mathbb{H}^n)$  act on  $R_\Gamma(n)$  by conjugation, and we will consider the quotients  $R_\Gamma(n)/\text{Isom}(\mathbb{H}^n)$  and  $R_\Gamma(n)/\text{Isom}^+(\mathbb{H}^n)$ , equipped with their quotient topologies.

We will mainly focus on the case when  $\Gamma = \pi_1\Sigma_g$  is the fundamental group of a closed surface of genus  $g$ , with a given standard presentation (i.e., a *marking*); we then denote  $R_g(n) = R_{\pi_1\Sigma_g}(n)$ . Also, we are mainly interested in the case  $n = 2$  (thus  $\text{Isom}^+(\mathbb{H}^2) = \text{PSL}(2, \mathbb{R})$ ), and we denote  $R_g = R_g(2)$ .

The space  $R_g$  is a real algebraic variety (see [CS83]), and is smooth, of dimension  $6g - 3$ , outside of the set of abelian representations (see [Gol84]). Outside of the set of (classes of) *elementary* representations (i.e., having a global fixed point in  $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ ), the quotient  $R_g/\text{PSL}(2, \mathbb{R})$ , equipped with the quotient topology, is again smooth of dimension  $6g - 6$ ; one of its connected components is naturally identified with the Teichmüller space of the surface (see [Gol80, Gol88]).

A function  $e: R_g \rightarrow \mathbb{Z}$ , called the *Euler class*, plays a key role in understanding the spaces  $R_g$  and  $R_g/\text{PSL}(2, \mathbb{R})$ . There are several ways to define it; for instance, a representation  $\rho \in R_g$  defines a circle bundle on  $\Sigma_g$ , which has a  $\mathbb{Z}$ -valued characteristic class, the Euler class. We will give complete recalls on this Euler class in section 2.3, and refer to [Mil58, Woo71, Ghy87a, Mat87, Gol88, Cal04] for a deep understanding of this class.

In 1988, W. Goldman proved [Gol88] that  $R_g$  possesses  $4g - 3$  connected components, which are the preimages  $e^{-1}(k)$ , for  $2 - 2g \leq k \leq 2g - 2$  (the fact that the absolute value of the Euler class is bounded by  $2g - 2$  was previously known as the Milnor-Wood inequality, see [Mil58, Woo71]). He proved, moreover, that the components  $e^{-1}(2 - 2g)$  and  $e^{-1}(2g - 2)$  consist precisely in *all* the discrete and faithful representations of  $\pi_1\Sigma_g$  into  $\text{PSL}(2, \mathbb{R})$ . He also proved that the space  $R_g(3) = \text{Hom}(\pi_1\Sigma_g, \text{PSL}(2, \mathbb{C}))$  has two connected components. The elements of  $R_g(2)$  of even Euler class, on one hand, and the ones of odd Euler class on the other hand, fall into these two different components of  $R_g(3)$ .

The Euler class is still well-defined in the quotient space  $R_g/\text{PSL}(2, \mathbb{R})$  (we denote  $\mathbf{e}: R_g/\text{PSL}(2, \mathbb{R}) \rightarrow \mathbb{Z}$ ), whence the space  $R_g/\text{PSL}(2, \mathbb{R})$  also admits  $4g - 3$  connected components, similarly. However, only the absolute value of the Euler class is defined on  $R_g/\text{Isom}(\mathbb{H}^2)$ , which has  $2g - 1$  connected components. In  $R_g/\text{PSL}(2, \mathbb{R})$ , the connected component  $\mathbf{e}^{-1}(2 - 2g)$  is naturally identified with the Teichmüller space of  $\Sigma_g$ , and  $\mathbf{e}^{-1}(2g - 2)$  is the space of marked hyperbolic metrics of constant curvature  $-1$  on the surface  $\Sigma_g$  equipped with the opposite orientation. These consist in all the classes of discrete and faithful representations, given by the holonomies of such structures.

In 1976, W. Thurston [Thu88] introduced a natural compactification of Teichmüller spaces; his construction was intensively studied and detailed in [FLP91]. Recall that the Teichmüller space of the surface,  $\mathfrak{e}^{-1}(2 - 2g)$ , is the space of marked hyperbolic metrics on  $\Sigma_g$ , of constant curvature  $-1$ . A natural system of coordinates, on the Teichmüller space, consists in taking the lengths of closed geodesics on the surface. In that way, one proves that the Teichmüller space is homeomorphic to  $\mathbb{R}^{6g-6}$  (indeed, the lengths of  $6g - 6$  curves suffice to parameter hyperbolic metrics). W. Thurston's compactification of the Teichmüller space consists in embedding this space into a projective space (thus, considering lengths only up to a scalar), in which the Teichmüller space has relatively compact image. A very important feature of this compactification is that the boundary added to the Teichmüller space is homeomorphic to a sphere of dimension  $6g - 7$ , in such a way that the Thurston compactification of the Teichmüller space is homeomorphic to a closed ball of dimension  $6g - 6$ .

This compactification has been extended to other connected components of  $R_g/PSL(2, \mathbb{R})$ , and to other representation spaces in successive works. In 1984, J. Morgan and P. Shalen [MS84], using techniques of algebraic geometry, defined a compactification of the real algebraic variety  $X_{\Gamma, SL(2, \mathbb{R})}$  of characters of representations of  $\Gamma$  in  $SL(2, \mathbb{R})$ . We denote  $\overline{X_{\Gamma, SL(2, \mathbb{R})}}^{MS}$  this compactification, and  $\overline{X_{g, SL(2, \mathbb{R})}}^{MS}$  in the case  $\Gamma = \pi_1 \Sigma_g$ . Since all the representations of even Euler class in  $PSL(2, \mathbb{R})$  (and, in particular, elements of the Teichmüller space) lift to  $SL(2, \mathbb{R})$  (see e.g. [Gol88]), this defines a compactification of Teichmüller spaces, and it coincides with the one of W. Thurston's. In [Mor86], J. Morgan generalized this construction to the group  $SO(n, 1)$ , for  $n \geq 2$ . In 1988 [Bes88, Pau88], M. Bestvina and F. Paulin, independently, gave a much more geometric viewpoint to this compactification, for representations in  $Isom^+(\mathbb{H}^n)$ ,  $n \geq 2$ . F. Paulin proved that the quotient topology, on the space  $R_{\Gamma}(n)/Isom(\mathbb{H}^n)$ , coincides with the equivariant Gromov topology (which we will discuss in detail in the section 2.2). Equipped with this topology, the space  $m_{\Gamma}^{fd}(n)$  of (conjugacy classes of) faithful and discrete representations has a natural compactification, which recovers the compactification of [Mor86, MS84]. Note, finally, that A. Parreau extended [Par] this compactification to representation spaces in higher rank groups.

This compactification being well-defined, it is a natural question to ask if it respects the topology and geometry of  $R_g/PSL(2, \mathbb{R})$ , as does the Thurston compactification, when restricted to the Teichmüller space alone. We shall prove that it is not the case, and that this compactification, as defined in all the works mentioned above, leads to a very degenerate space.

The works of J. Morgan and P. Shalen, of M. Bestvina and of F. Paulin all yield the same compactification, and we shall follow the approach of F. Paulin, in which it is easier to add a notion of orientation. In order to use F. Paulin's construction, we first define *explicitly* the biggest Hausdorff quotients of  $R_{\Gamma}(n)/Isom(\mathbb{H}^n)$  and  $R_{\Gamma}(n)/Isom^+(\mathbb{H}^n)$ , that we denote  $m_{\Gamma}^u(n)$  and  $m_{\Gamma}^o(n)$ , respectively (indeed, these spaces are not Hausdorff in general, so that M. Bestvina and F. Paulin's constructions cannot be extended literally to

these spaces). Again, we write  $m_\Gamma^u = m_\Gamma^u(2)$ ,  $m_g^u(n) = m_{\pi_1 \Sigma_g}(n)$  and so on. Note that, in the space  $m_g^u$ , only the absolute value of the Euler class is still defined, and  $m_g^u$  possesses  $2g - 1$  connected components. We denote by  $\overline{m_g^u(n)}$  the compactification of  $m_g^u(n)$  as it is constructed by F. Paulin.

Also, in all this text, for all  $k \in \{2 - 2g, \dots, 2g - 2\}$ , we will denote by  $m_{g,k}^o$  the subset of  $m_g^o$  consisting of classes of representations of Euler class  $k$ , and for  $k \in \{0, \dots, 2g - 2\}$ ,  $m_{g,k}^u$  will denote the connected component of  $m_g^u$  consisting of classes of representations whose Euler class, in absolute value, equals  $k$ .

**Theorem 1.1** *Let  $g \geq 4$  and  $k \in \{0, \dots, 2g - 3\}$ . Then, in  $\overline{m_g^u}$ , the boundary of the Teichmüller space,  $\partial m_{g,2g-2}^u$ , is contained in  $\partial m_{g,k}^u$  as a closed, nowhere dense subset.*

In particular,

**Corollary 1.2** *For all  $g \geq 4$ ,  $\overline{m_g^u(2)}$  is connected.*

We also prove that for  $g = 2$  and  $g = 3$ , the space  $\overline{m_g^u(2)}$  possesses at most two connected components. Actually, theorem 1.1 should hold for all  $g \geq 2$ , but in the case where  $k = 1$  the proofs presented here request that  $g \geq 4$ .

In particular, the two connected components of  $Hom(\pi_1 \Sigma_g, PSL(2, \mathbb{C})) / PSL(2, \mathbb{C})$  meet at their boundary in this compactification, as soon as  $g \geq 3$ :

**Corollary 1.3** *For all  $g \geq 3$ , the space  $\overline{m_g^u(3)}$  is connected.*

If we consider only representations of even Euler class (i.e., that lift to  $SL(2, \mathbb{R})$ ) then the result holds for all  $g \geq 2$ :

**Corollary 1.4** *For all  $g \geq 2$ , the space  $\overline{X_{g,SL(2,\mathbb{R})}}^{MS}$  is connected.*

Theorem 1.1 not only implies that the space  $\overline{m_g^u}$  is connected, but it also implies that this space is particularly degenerated. Since the connected components  $m_{g,k}^u$  are of dimension  $6g - 6$  (see [Gol84], section 1), one should expect that the boundary  $\partial m_{g,k}^u$  has dimension  $6g - 7$ . However, the boundary of the Teichmüller space itself has dimension  $6g - 7$  (see e.g. [FLP91], exposé 1, theorem 1). Therefore, it follows from theorem 1.1 that the compactifications  $\overline{m_{g,k}^u}$  of the “exotic” connected components (i.e., the connected components  $m_{g,k}^u$  such that  $|k| \leq 2g - 3$ ) have no PL structures, or cell complex structures or so on, compatible with the compactification. This contrasts very strongly with the behaviour of the Thurston compactification of the Teichmüller spaces.

The proof of theorem 1.1 uses the following fact, interesting for itself:

**Proposition 1.5** *For all  $k$  such that  $|k| \leq 2g - 2$ , the connected components  $m_{g,k}^o$  and  $m_{g,|k|}^u$  are one-ended (i.e., all the connected components of  $m_g^o$  and  $m_g^u$  are one-ended).*

Actually, for all  $k \neq 0$ , this follows from a theorem of N. Hitchin (see [Hit87], proposition 10.2), which says that the connected component  $m_{g,k}^u$  is homeomorphic to a complex vector bundle over the  $(2g - 2 - |k|)$ -th symmetric product of the surface  $\Sigma_g$ . However, the proof of proposition 1.5 is far more simple and extends to the case  $k = 0$ .

The degeneration of the compactification of  $m_g^u$  is due (at least) to the fact that the equivariant Gromov topology forgets the orientation of the space  $\mathbb{H}^n$ . This information, in the case  $n = 2$ , is the information carrying the Euler class, and which separates the space  $m_g^u$  into its connected components. By restauring this orientation we will define a new compactification of these representation spaces, cancelling (partly, at least) this degeneration.

We define a notion of convergence in the sense of Gromov for *oriented* spaces, which preserves the orientation. This enables us to define a new compactification of  $m_\Gamma^o$ , in which the ideal points, added at the boundary, are *fat*  $\mathbb{R}$ -trees, that is,  $\mathbb{R}$ -trees together with an “orientation”. These fat  $\mathbb{R}$ -trees form a set  $\mathcal{T}^o(\Gamma)$ , which compactifies the space  $m_\Gamma^o$ :

**Theorem 1.6** *The map  $m_\Gamma^o \rightarrow m_\Gamma^o \cup \mathcal{T}^o(\Gamma)$  induces a natural compactification of  $m_\Gamma^o$ . Moreover, the natural map  $\pi : \overline{m_\Gamma^o} \rightarrow \overline{m_\Gamma^u}$ , which consists in forgetting the orientation, is onto, and some of its fibres have the same cardinality as  $\mathbb{R}$ .*

By *natural*, we mean, following F. Paulin [Pau04], that the action of the mapping class group extends continuously to the compact space  $\overline{m_g^o}$ .

One can define an Euler class in a quite general context, as we will see in the section 2.3. In particular, the actions of  $\pi_1 \Sigma_g$  on fat  $\mathbb{R}$ -trees admit an Euler class, and we shall prove the following:

**Theorem 1.7** *The Euler class  $\mathbf{e} : \overline{m_g^o} \rightarrow \mathbb{Z}$  is a continuous function. In particular, the compactification  $\overline{m_g^o}$  possesses as many connected components as the space  $m_g^o$ .*

In theorem 1.6, the fact that some fibres of the map  $\pi$  have the same cardinality as  $\mathbb{R}$  will depend on the following remarkable property of rigidity of non discrete representations of  $\pi_1 \Sigma_g$  in  $PSL(2, \mathbb{R})$ . Let  $o$  denote the natural cyclic order on  $\partial \mathbb{H}^2 = \mathbb{S}^1$ , i.e., for all  $a, b, c \in \mathbb{S}^1$ , put  $o(a, b, c) = 0$  if two of these three points coincide, 1 if  $a, b, c$  are met in the anticlockwise order on the circle, and  $-1$  otherwise.

**Proposition 1.8** *Let  $\rho_1, \rho_2 \in R_g$  be two non discrete, non elementary representations, and let  $a_1, a_2 \in \partial \mathbb{H}^2$ . Suppose that for all  $\gamma_1, \gamma_2, \gamma_3 \in \pi_1 \Sigma_g$ ,*

$$o(\rho_1(\gamma_1) \cdot a_1, \rho_1(\gamma_2) \cdot a_1, \rho_1(\gamma_3) \cdot a_1) = o(\rho_2(\gamma_1) \cdot a_2, \rho_2(\gamma_2) \cdot a_2, \rho_2(\gamma_3) \cdot a_2).$$

*Then there exists  $g \in PSL(2, \mathbb{R})$  such that  $\rho_1 = g\rho_2g^{-1}$ .*

In other words, two representations are conjugate if and only if they have orbits which are isomorphic as cyclically ordered sets. It follows that two non elementary, non discrete representations are conjugate by an element of

$PSL(2, \mathbb{R})$  if and only if they are conjugate by an element of  $Homeo^+(\mathbb{S}^1)$ , as representations in  $Homeo^+(\mathbb{S}^1)$ . Since all the orbits of these representations are dense in  $\mathbb{S}^1$ , it follows from proposition 2-2 of E. Ghys [Ghy87b] that two non elementary, non discrete representations are conjugate in  $PSL(2, \mathbb{R})$  if and only if they are topologically semi-conjugate, which is equivalent to saying that they define the same element in the bounded cohomology group  $H_b^2(\pi_1 \Sigma_g, \mathbb{Z})$  (see [Ghy87b]).

This contrasts drastically with the case of representations of maximal Euler class. A theorem of S. Matsumoto [Mat87] states that if two representations of  $\pi_1 \Sigma_g$  in  $Homeo^+(\mathbb{S}^1)$  both have Euler class  $2g - 2$  then they are conjugate. In other words, in the space  $Hom(\pi_1 \Sigma_g, Homeo^+(\mathbb{S}^1))/Homeo^+(\mathbb{S}^1)$ , the subspace  $\mathbf{e}^{-1}(2g-2)$  is a single point, whereas  $\mathbf{e}^{-1}(k)$ , for  $|k| \leq 2g-3$ , contains the injective image of a space of dimension  $6g - 6$  (indeed, it is proved in [FW07], proposition 4.4, see also [Wol07], proposition 4.1.15) that the space of non elementary, non discrete representations is an open and dense subspace of each connected component of  $Hom(\pi_1 \Sigma_g, PSL(2, \mathbb{R}))/PSL(2, \mathbb{R})$  of non extremal Euler class.

By theorem 1.6, it is a necessary condition, for an action of  $\pi_1 \Sigma_g$  on an  $\mathbb{R}$ -tree to be in  $\overline{m_g^u}$ , to preserve some orientation. In particular, we can construct explicit actions on  $\mathbb{R}$ -trees which are not the limit of actions of surface groups on  $\mathbb{H}^2$ . These explicit actions can even be obtained as limits of actions of surface groups on  $\mathbb{H}^3$ , so that we get the following proposition:

**Proposition 1.9** *Let  $g \geq 3$ . There exist minimal actions of  $\pi_1 \Sigma_g$  on  $\mathbb{R}$ -trees by isometries, which are in  $\overline{\partial m_g^u(3)}$  but not in  $\overline{\partial m_g^u(2)}$ .*

This contrasts severely with the case of discrete and faithful representations. Indeed, R. Skora [Sko96] proved that a minimal action of  $\pi_1 \Sigma_g$  on an  $\mathbb{R}$ -tree has small arc stabilizers (i.e., the stabilizer of any pair of distinct points of the tree is virtually abelian) if and only if it is the limit of discrete and faithful representations of  $\pi_1 \Sigma_g$  in  $PSL(2, \mathbb{R})$  (or equivalently, it is a point in the boundary of the Teichmüller space), and it is well-known (see e.g. [MS84, Pau88, Bes88]) that limits of discrete and faithful representations in  $Isom^+(\mathbb{H}^n)$  enjoy that property. As a corollary of R. Skora's result, we thus have  $\overline{\partial m_g^{fd}(n)} = \overline{\partial m_g^{fd}(2)}$ . When we consider representations which may not be faithful and discrete, proposition 1.9 states that this equality does not hold any more.

Note that C. McMullen has developed, independently, a theory of oriented  $\mathbb{R}$ -trees, under the name of *ribbon  $\mathbb{R}$ -trees*, in [McM], in order to compactify the set of proper holomorphic maps from the unit disk  $\Delta \subset \mathbb{C}$  into itself.

This text is organized as follows. Section 2 gathers every background material concerning the spaces that we wish to compactify. Section 2.1 is devoted to the explicit construction of  $m_\Gamma^u(n)$  and  $m_\Gamma^o(n)$ , the biggest Hausdorff quotients of  $R_\Gamma(n)/Isom(\mathbb{H}^n)$  and  $R_\Gamma(n)/Isom^+(\mathbb{H}^n)$ . In section 2.2, we give

some recalls on F. Paulin's point of view on the compactification of  $m_\Gamma^u(n)$ , while adapting it slightly so that it indeed defines a compactification of the whole space  $m_\Gamma^u(n)$ . In section 2.3, we recall the construction of the Euler class, and establish a technical lemma which will enable us to prove that the Euler class extends continuously to the boundary (in the oriented compactification) of  $m_g^o$  (theorem 1.7). Section 2.4 is devoted to the proof of proposition 1.8. Section 2.5 recalls an argument of Z. Sela in the context of limit groups, which plays a key role in the proof of theorem 1.1. Section 3 contains the core of our results on the compactification. It begins with the proof of proposition 1.5, then turns to the degenerations of  $\overline{m_g^u}$ , and finally we construct the oriented compactification.

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## 2 Preliminaries

### 2.1 The representation spaces

Let us first remark that the quotient spaces  $m_\Gamma^u(n) = R_\Gamma(n)/Isom^+(\mathbb{H}^n)$  and  $R_\Gamma(n)/Isom(\mathbb{H}^n)$  are not Hausdorff in general. Indeed, in  $Isom^+(\mathbb{H}^2) = PSL(2, \mathbb{R})$ , the matrix  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  is conjugate to  $\begin{pmatrix} 1 & \frac{1}{t^2} \\ 0 & 1 \end{pmatrix}$ , for all  $t \in \mathbb{R}^*$ , hence its conjugacy class cannot be separated from the one of the identity. As soon as there exists a morphism of  $\Gamma$  onto  $\mathbb{Z}$ , we can therefore construct abelian representations of  $\Gamma$  in  $PSL(2, \mathbb{R})$ , and more generally in  $Isom^+(\mathbb{H}^n)$  (which contains isomorphic copies of  $PSL(2, \mathbb{R})$ ), which are not separated from the trivial representation in these quotient spaces.

However, in order to use F. Paulin's construction of the compactification of these representation spaces, we need to work with Hausdorff spaces.

Most of the arguments presented in this section can be expressed in a more algebraic manner, and can be generalized to every semi-simple Lie groups (see [Par]). The construction we propose here uses only tools of hyperbolic geometry.

In order to set up the notations, we first make some recalls on the real hyperbolic space  $\mathbb{H}^n$  and its group of orientation-preserving isometries. We will be using the hyperboloid model, and we refer to [BP92], Chapter A, for a complete overview).



Equip the space  $\mathbb{R}^{n+1}$  with the quadratic form

$$q(x_0, x_1, \dots, x_n) = x_0 x_1 + \frac{1}{2} (x_2^2 + \dots + x_n^2).$$

The set  $\{\underline{x} \in \mathbb{R}^{n+1} | q(\underline{x}) = -1\}$  (where  $\underline{x}$  stands for  $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ ) has two connected components, and the set  $\mathbb{H}^n$  is defined as

$$\mathbb{H}^n = \{\underline{x} \in \mathbb{R}^{n+1} | q(\underline{x}) = -1, x_1 - x_0 > 0\}.$$

The form  $\langle \underline{x}, \underline{y} \rangle = x_0 y_1 + x_1 y_0 + x_2 y_2 + \dots + x_n y_n$  defines a scalar product on the tangent space at each point of  $\mathbb{H}^n$ , hence a Riemannian metric on  $\mathbb{H}^n$ . The geodesics of this space are its intersections with the (vectorial) planes of  $\mathbb{R}^{n+1}$ .

Denote by  $SO_{\mathbb{R}}(n, 1)$  the subgroup of  $SL(n+1, \mathbb{R})$  consisting of elements preserving  $q$ . Denote by  $SO_{\mathbb{R}}^+(n, 1)$  the index 2 subgroup of  $SO_{\mathbb{R}}(n, 1)$  formed by elements preserving  $\mathbb{H}^n$  (the other elements exchange the two connected components of  $q^{-1}(-1)$ ). Then  $SO_{\mathbb{R}}^+(n, 1)$  is the group of orientation-preserving isometries of  $\mathbb{H}^n$ .

The image of  $\mathbb{H}^n$  on the hyperplane  $\{x_0 - x_1 = 0\}$ , under the stereographic projection of center  $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \dots, 0\right)$  is an open disk of centre 0 and radius 1; we denote it  $\mathbb{D}^n$ . This yields the usual compactification of  $\mathbb{H}^n$ , and the projection of any geodesic of  $\mathbb{H}^n$  on  $\mathbb{D}^n$  is a geodesic in  $\mathbb{D}^n$  for the Poincaré metric.

In  $SO_{\mathbb{R}}^+(n, 1)$ , the stabilizer of the point  $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \dots, 0\right) \in \partial \mathbb{D}^n$  is the subgroup of  $SO_{\mathbb{R}}^+(n, 1)$  formed by matrices of the form

$$\begin{pmatrix} \lambda & 0 & (0) \\ r & \frac{1}{\lambda} & Y \\ Z & (0) & A \end{pmatrix},$$

with  $\lambda > 0$ ,  $A \in SO(n-1)$ ,  $\|Y\|^2 = -\frac{2r}{\lambda}$  and  $A^t Y = -\frac{1}{\lambda} Z$ . The subgroup of elements which also fix the point  $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0\right) \in \partial \mathbb{D}^n$  (and in particular, which preserve globally the geodesic  $\mathbb{H}^n \cap \{x_2 = \dots = x_n = 0\}$ ) is formed by matrices of the form

$$\begin{pmatrix} \lambda & 0 & (0) \\ 0 & \frac{1}{\lambda} & (0) \\ (0) & (0) & A \end{pmatrix},$$

with  $\lambda > 0$  et  $A \in SO(n-1)$ . By analogy with the classical case  $n = 2$ , and with the model of the upper half plane, we denote by 0 the point  $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \dots, 0\right) \in \partial \mathbb{D}^n$ , and the point  $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0\right) \in \partial \mathbb{D}^n$  is denoted  $\infty$ .

Elements of  $SO_{\mathbb{R}}^+(n, 1)$  having a fixed point in  $\mathbb{H}^n$  are called *elliptic*, elements with a unique fixed point in  $\partial \mathbb{H}^n$  are termed *parabolic*, and elements  $\varphi \in SO_{\mathbb{R}}^+(n, 1)$  such that  $\inf_{x \in \mathbb{H}^n} d(x, \varphi(x)) > 0$  (this lower bound is then

achieved) are called *loxodromic*. Loxodromic isometries fix two points in  $\partial\mathbb{H}^n$ , and are conjugate to a matrix of the form

$$\begin{pmatrix} \lambda & 0 & (0) \\ 0 & \frac{1}{\lambda} & (0) \\ (0) & (0) & A \end{pmatrix}.$$

When  $A = \text{Id}$ , we also say that  $\varphi$  is *hyperbolic*.

In some texts (which do not include this paper), in the case  $n = 2$ , the identity is not considered as an elliptic element.

The lower bound

$$d(u) = \inf_{x \in \mathbb{H}^n} d(x, u \cdot x)$$

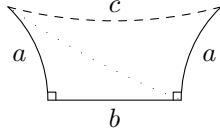
is achieved if and only if  $u$  is non parabolic. In that case we denote by  $\min(u)$  the set  $\{x \in \mathbb{H}^n \mid d(x, u \cdot x) = d(u)\}$ , and if  $r > 0$ , the set  $\{x \in \mathbb{H}^n \mid d(x, u \cdot x) < r + d(u)\}$  is denoted  $\min_r(u)$ .

The following lemma will be necessary to prove the local compactness of the quotients  $m_\Gamma^o(n)$  and  $m_\Gamma^u(n)$ .

**Lemma 2.1** *Let  $u \in \text{Isom}^+(\mathbb{H}^n)$  be a non parabolic element. Then  $\min_r(u)$  is at bounded distance of  $\min(u)$ . More explicitly, for all  $r > 0$ , there exists  $k > 0$  such that for all  $x \in \min_r(u)$  we have  $d(x, \min(u)) < k$ .*

**Proof**

First suppose that  $u$  is loxodromic. In the following figure in the hyperbolic plane, the cosine law I gives the formula



$$\cosh c = 1 + \cosh^2 a (\cosh b - 1).$$

In particular, for all  $x \in \mathbb{H}^n$  we have

$$\cosh(d(x, u \cdot x)) \geq 1 + (\cosh(d(u)) - 1) \cdot \cosh^2(d(x, \min(u))),$$

hence the lemma, in that case.

Now suppose that  $u$  fixes a unique point  $x_0$  in  $\mathbb{H}^n$ . Then the angle between  $[x_0, x]$  and  $[x_0, ux]$ , as  $x$  describes the sphere of radius 1 around  $x_0$ , achieves a minimum  $\theta_0 \neq 0$ . For all  $x \in \mathbb{H}^n \setminus \{x_0\}$ , denote by  $\theta(x)$  the angle, at  $x_0$ , in the triangle  $\Delta(x, x_0, u \cdot x)$ . The cosine law I in this triangle then gives

$$\begin{aligned} \cosh(d(x, u \cdot x)) &= 1 + \sinh^2(d(x, \min(u)))(1 - \cos \theta) \\ &\geq 1 + \sinh^2(d(x, \min(u)))(1 - \cos \theta_0), \end{aligned}$$

which proves the lemma in this case.

Suppose finally that  $u$  is elliptic, and fixes pointwise (at least) a geodesic in  $\mathbb{H}^n$ . Choose a point  $x_0 \in \min(u)$ . For all  $x \in \mathbb{H}^n$ , denote  $\pi(x)$  the projection

of  $x$  on  $\min(u)$ . Up to conjugating by an isometry which commutes with  $u$ , we may suppose that  $\pi(x)$  is the point  $x_0$ . Then, as  $x$  describes the compact set  $\{x \in \mathbb{H}^n \mid \pi(x) = x_0, d(x, x_0) = 1\}$ , the angle between the segments  $[x_0, x]$  and  $[x_0, \pi(x)]$  achieves a minimum  $\theta_0 \neq 0$ . One finds once again, as in the preceding case, for all  $x \in \mathbb{H}^n$ , the formula

$$\cosh(d(x, u \cdot x)) \geq 1 + \sinh^2(d(x, \min(u)))(1 - \cos \theta_0)$$

which completes the proof.  $\square$

Also, we shall use the following classical property:

**Proposition 2.2** *For every  $x \in \mathbb{H}^n$ , and every  $d \in \mathbb{R}$ , the set*

$$\{\gamma \in \text{Isom}(\mathbb{H}^n) \mid d(x, \gamma x) \leq d\}$$

*is compact.*

Now we can define the space  $m_\Gamma^\circ(n)$ :

**Definition 2.3** *We denote by  $m_\Gamma^\circ(n)$  the subspace of  $m'_\Gamma(n)$  formed by classes of representations which have either 0, or at least 2 global fixed points in  $\partial\mathbb{H}^n$ .*

We have an application  $i: m_\Gamma^\circ(n) \hookrightarrow m'_\Gamma(n)$ . We can also define an application  $\pi: m'_\Gamma(n) \twoheadrightarrow m_\Gamma^\circ(n)$  as follows. If  $c \in m_\Gamma^\circ(n)$ , put  $\pi(c) = c$  (in particular,  $\pi$  is onto). If  $c = [\rho] \in m'_\Gamma(n) \setminus m_\Gamma^\circ(n)$ , then  $\rho$  possesses a unique fixed point  $r_1 \in \partial\mathbb{H}^n$ . Choose another point  $r_2 \in \partial\mathbb{H}^n \setminus \{r_1\}$  arbitrarily, and denote by  $g_k \in SO_{\mathbb{R}}^+(n, 1)$  the hyperbolic isometry of axis  $(r_1, r_2)$ , and attractive point  $r_1$ , with translation distance  $k$ .

**Lemma 2.4** *The sequence  $(g_k^{-1} \rho g_k)_{k \in \mathbb{N}}$  converges to a representation  $\rho_\infty \in R_\Gamma(n)$  such that  $[\rho_\infty] \in m_\Gamma^\circ(n)$ , and such that  $[\rho_\infty]$  does not depend on the choice of  $\rho$  in the conjugacy class  $c$  either of the choice of  $r_2$ .*

We can therefore set  $\pi(c) = [\rho_\infty]$ .

**Proof**

Choose a representant  $\rho$  of the conjugacy class  $c$  so that  $\rho$  fixes  $0 \in \partial\mathbb{H}^n$  (in other words, conjugate  $\rho$  by an isometry sending  $r_1$  to 0). Take  $r_2 = \infty$ . Then for all  $\gamma \in \Gamma$ ,  $\rho(\gamma)$  is of the form

$$\rho(\gamma) = \begin{pmatrix} \lambda(\gamma) & 0 & (0) \\ r(\gamma) & \frac{1}{\lambda(\gamma)} & Y(\gamma) \\ Z(\gamma) & (0) & A(\gamma) \end{pmatrix}.$$

In this basis,  $g_k$  has the form

$$g_k = \begin{pmatrix} t_k & & (0) \\ & \frac{1}{t_k} & \\ (0) & & \text{I}_{n-1} \end{pmatrix},$$

where  $t_k \rightarrow 0$  as  $k \rightarrow +\infty$ . Then  $g_k^{-1}\rho g_k$  converges to the representation  $\rho_\infty$  such that for all  $\gamma \in \Gamma$ ,

$$\rho_\infty(\gamma) = \begin{pmatrix} \lambda(\gamma) & & (0) \\ & \frac{1}{\lambda(\gamma)} & \\ (0) & & A(\gamma) \end{pmatrix},$$

which indeed fixes the points 0 and  $\infty$  at the boundary.

Choosing another representant  $\rho'$  of  $c$  fixing 0 would simply be the same as considering a conjugate  $\rho' = h^{-1}\rho h$ , where  $h \in SO_{\mathbb{R}}^+(n, 1)$  has the form

$$h = \begin{pmatrix} \lambda & 0 & (0) \\ r & \frac{1}{\lambda} & Y \\ Z & (0) & A \end{pmatrix},$$

and conjugation by  $h$  does not touch the elements  $\lambda(\gamma)$  and  $A(\gamma)$ , which determine the representation  $\rho_\infty$ . Finally, the choice of another  $r_2$  amounts to conjugate  $\rho$  by an orientation-preserving isometry of  $\mathbb{H}^n$  fixing 0; we have just dealt with this case.  $\square$

It follows that we can equip the set  $m_\Gamma^o(n)$  both with the induced topology, and of the quotient topology determined by the surjection  $\pi: m'_\Gamma(n) \twoheadrightarrow m_\Gamma^o(n)$ . The object of this section is to prove the following.

**Theorem 2.5** *The induced topology and the quotient topology coincide on  $m_\Gamma^o(n)$  (in particular,  $\pi$  is continuous). Moreover, the space  $m_\Gamma^o(n)$  is Hausdorff, and locally compact.*

In particular, this topology is also the quotient topology corresponding to the surjection  $R_\Gamma(n) \twoheadrightarrow m_\Gamma^o(n)$ .

It follows that  $m_\Gamma^o(n)$  is the biggest Hausdorff quotient of  $m'_\Gamma(n)$ , in the following sense:

**Definition 2.6** *Let  $X$  be a topological space. A quotient space  $\pi: X \rightarrow X_s$  is called the biggest Hausdorff quotient of  $X$  if  $X_s$  is Hausdorff and if for every continuous mapping  $f: X \rightarrow Y$  to a Hausdorff space  $Y$ , there exists a unique function  $\bar{f}: X_s \rightarrow Y$  such that  $f = \pi \circ \bar{f}$ .*

In particular, such a quotient, if it exists, is unique, up to *canonical* homeomorphism. Clearly, every Hausdorff space is itself its biggest Hausdorff quotient.

The representations which have no fixed points in  $\partial\mathbb{H}^n$  are called *non parabolic*, and in the sequel of this section, everything that concerns only non parabolic representations is already written in [Par], in the much more general context of actions on  $CAT(0)$  spaces.

We first prove that non parabolic representations form an open subset of  $m'_\Gamma(n)$ :

**Lemma 2.7 (compare with [Par], proposition 2.6)** *Let  $\rho \in R_\Gamma(n)$  be non parabolic. Then  $[\rho]$  possesses, in  $m'_\Gamma(n)$ , a neighbourhood consisting of non parabolic representations.*

**Proof**

The space  $R_\Gamma(n) \subset (M_{n+1}(\mathbb{R}))^S$  is a metric space, and the map  $R_\Gamma(n) \rightarrow m'_\Gamma(n)$  is open, hence the space  $m'_\Gamma(n)$  is locally numerable (every point possesses a numerable fundamental system of neighbourhoods); it follows that we can use sequential criteria in this space.

Consider  $(\rho_k)_k \in m'_\Gamma(n)^\mathbb{N}$  such that for all  $k$ ,  $\rho_k$  has a fixed point  $r_k \in \partial\mathbb{H}^n$ ; and suppose that  $\rho_k \rightarrow \rho$ : let us prove that  $\rho$  has a fixed point in  $\partial\mathbb{H}^n$ . Up to extract a subsequence,  $r_k$  converges to a point  $r \in \partial\mathbb{H}^n$ . Then there exist  $h_k \in SO_\mathbb{R}^+(n, 1)$  such that  $h_k(r_k) = r$  and such that  $h_k \rightarrow \text{Id}$ . Then  $h_k \rho_k h_k^{-1}$  fixes  $r$  globally, and converges to  $\rho$  hence  $\rho$  fixes  $r$  globally. It follows that the set of representations which have at least one fixed point in  $\partial\mathbb{H}^n$  is a closed subset of  $m'_\Gamma(n)$ .  $\square$

The first step towards the continuity of  $\pi$  is the following.

**Lemma 2.8** *Let  $(\rho_k)_k$  be a sequence of representations, each having a unique fixed point in  $\partial\mathbb{H}^n$ , converging to a representation  $\rho$  which has at least two fixed points in  $\partial\mathbb{H}^n$ . Then  $\pi([\rho_k])$  converges to  $[\rho]$ , in the space  $m'_\Gamma$ .*

**Proof**

Let us first prove that up to considering a subsequence, the sequence  $\pi([\rho_k])$  converges to  $[\rho]$ . The fixed point  $r_k$  of  $\rho_k$  stays in the compact space  $\partial\mathbb{H}^n$ , hence there is a subsequence  $r_{\varphi(k)}$  of fixed points of  $\rho_{\varphi(k)}$  which converges to a point  $r \in \partial\mathbb{H}^n$ , which is therefore a fixed point of  $\rho$ . For all  $k$ , there exists an orientation-preserving isometry  $h_{\varphi(k)}$  of  $\mathbb{H}^n$  such that  $h_{\varphi(k)}(r_{\varphi(k)}) = r$ , satisfying  $h_{\varphi(k)} \rightarrow \text{Id}$ . Then  $h_{\varphi(k)}^{-1} \rho_{\varphi(k)} h_{\varphi(k)}$  converges to  $\rho$ , and fixes  $r$ . In other words, we may suppose that  $\rho_{\varphi(k)}$  fixes  $r$ , and up to conjugation we can further suppose that  $r = 0$ . Now for all  $\gamma \in \Gamma$ ,  $\rho_{\varphi(k)}(\gamma)$  and  $\rho(\gamma)$  are under the following form:

$$\rho_{\varphi(k)}(\gamma) = \begin{pmatrix} \lambda_{\varphi(k)}(\gamma) & 0 & (0) \\ r_{\varphi(k)}(\gamma) & \frac{1}{\lambda_{\varphi(k)}(\gamma)} & Y_{\varphi(k)}(\gamma) \\ Z_{\varphi(k)}(\gamma) & (0) & A_{\varphi(k)}(\gamma) \end{pmatrix}, \quad \rho(\gamma) = \begin{pmatrix} \lambda(\gamma) & 0 & (0) \\ 0 & \frac{1}{\lambda(\gamma)} & (0) \\ (0) & (0) & A(\gamma) \end{pmatrix},$$

and by construction, a representant (denote it  $\pi\rho_{\varphi(k)}$ ) of the conjugacy class  $\pi([\rho_{\varphi(k)}])$  has the form:

$$\pi\rho_{\varphi(k)}(\gamma) = \begin{pmatrix} \lambda_{\varphi(k)}(\gamma) & 0 & (0) \\ 0 & \frac{1}{\lambda_{\varphi(k)}(\gamma)} & (0) \\ (0) & (0) & A_{\varphi(k)}(\gamma) \end{pmatrix}.$$

Moreover,  $\rho_{\varphi(k)}(\gamma) \rightarrow \rho(\gamma)$  for all  $\gamma \in \Gamma$ , hence  $\lambda_{\varphi(k)}(\gamma) \rightarrow \lambda(\gamma)$  and  $A_{\varphi(k)}(\gamma) \rightarrow A(\gamma)$ , so that  $\pi\rho_{\varphi(k)} \rightarrow \rho$ .

The proof we have just given works for every subsequence of the sequence  $(\rho_k)_k$ . In particular, every subsequence of  $(\pi([\rho_k]))_k$  possesses a subsequence converging to  $[\rho]$ . This implies that  $\pi([\rho_k])$  converges to  $[\rho]$ .  $\square$

We now use an argument of M. Besvina ([Bes88], proposition 1.2).

**Lemma 2.9 (compare with [Par], proposition 2.5)** *Let  $[\rho] \in m_1^o(n)$ . Then the minimum*

$$\min_{x \in \mathbb{H}^n} \max_{s \in S} d(x, \rho(s) \cdot x)$$

*is achieved.*

**Proof**

Consider a minimizing sequence  $(x_n)_{n \in \mathbb{N}}$  for this number. If  $x_k$  leaves every compact subset of  $\mathbb{H}^n$ , then up to considering a subsequence,  $x_n$  converges to a boundary point  $r \in \partial\mathbb{H}^n$ . In that case,  $r$  is a global fixed point of  $\rho$ , and since  $[\rho] \in m_1^o(n)$ , there exists at least one other. Hence  $\rho$  fixes (globally) a geodesic line of  $\mathbb{H}^n$ , and acts by translations on this geodesic, and then every point of this geodesic achieves the minimum. On the other hand, if  $(x_k)_k$  is bounded, then it has a subsequence converging to some point  $x_\infty \in \mathbb{H}^n$ , which realizes this minimum.  $\square$

In the sequel, for every  $\rho$  such that  $[\rho] \in m_1^o(n)$ , we put

$$d(\rho) = \min_{x \in \mathbb{H}^n} \max_{s \in S} d(x, \rho(s) \cdot x),$$

$$\min(\rho) = \left\{ x \in \mathbb{H}^n \mid \max_{s \in S} d(x, \rho(s) \cdot x) = d(\rho) \right\}$$

and

$$\min_\varepsilon(\rho) = \left\{ x \in \mathbb{H}^n \mid \max_{s \in S} d(x, \rho(s) \cdot x) < d(\rho) + \varepsilon \right\}.$$

**Lemma 2.10** *Let  $\rho \in m_1^o(n)$ . If  $\rho$  fixes at least one point at the boundary, then  $\min(\rho)$  is the convex hull of the fixed points of  $\rho$  in  $\partial\mathbb{H}^n$ ; this is a totally geodesic subspace of  $\mathbb{H}^n$ . Otherwise,  $\min(\rho)$  is compact. In every case,  $\min_\varepsilon(\rho)$  is at bounded distance to  $\min(\rho)$ ; that is, for all  $\varepsilon > 0$ , there exists  $k > 0$  such that for all  $x \in \min_\varepsilon(\rho)$ , we have  $d(x, \min(\rho)) < k$ .*

**Proof**

Suppose that  $\rho$  has no fixed points in  $\partial\mathbb{H}^n$ . If there existed  $\varepsilon > 0$  such that  $\min_\varepsilon(\rho)$  is not bounded, there would exist a sequence  $(x_n)_{n \in \mathbb{N}}$  of elements of  $\min_\varepsilon(\rho)$ , converging to a point  $x_\infty \in \partial\mathbb{H}^n$ , and then  $x_\infty$  would be a fixed point of  $\rho$ . Hence, for all  $\varepsilon > 0$ ,  $\min_\varepsilon(\rho)$  is bounded; moreover  $\min(\rho)$  is a closed subset of  $\mathbb{H}^n$ , hence compact; this finishes the proof in this case.

Now suppose that  $\rho$  has at least two distinct fixed points  $x_1, x_2$  in  $\partial\mathbb{H}^n$ . If  $d(\rho) \neq 0$ , then there exists  $s \in S$  such that  $\rho(s)$  is loxodromic, of axis  $(x_1, x_2)$ , and then  $\min(\rho) = \min(\rho(s)) = (x_1, x_2)$ ; so  $\rho$  does not have any other fixed

points in  $\mathbb{H}^n$ , and for all  $\varepsilon > 0$ ,  $\min_\varepsilon(\rho)$  lies at bounded distance of  $\min(\rho)$ , by lemma 2.1. If  $d(\rho) = 0$ , then  $\rho$  fixes pointwise at least the line  $(x_1, x_2)$ . Then

$$\min(\rho) = \{x \in \mathbb{H}^n \mid \forall s \in S, \rho(s) \cdot x = x\} = \bigcap_{s \in S} \min(\rho(s))$$

is an intersection of totally geodesic subspaces of  $\mathbb{H}^n$ , hence it is a totally geodesic subspace of  $\mathbb{H}^n$ ; and it is the convex hull of the fixed points of  $\rho$  in  $\partial\mathbb{H}^n$ . For all  $\varepsilon > 0$ , we then have  $\min_r(\rho) = \bigcap_{s \in S} \min_r(\rho(s))$ . By lemma 2.1, for all  $s \in S$ , the set  $\min_\varepsilon(\rho(s))$  is at bounded distance of the subspace  $\min(\rho)$ . One then checks by induction on  $Card(S)$  that  $\min_\varepsilon(\rho)$  is at bounded distance to  $\min(\rho)$ .  $\square$

This function  $d: m_\Gamma^o(n) \rightarrow \mathbb{R}_+$  is quite natural:

**Proposition 2.11** *The function  $d': R_\Gamma(n) \rightarrow \mathbb{R}_+$  defined by  $d'(\rho) = \inf_{x \in \mathbb{H}^n} \max_{s \in S} d(x, \rho(s)x)$  is continuous.*

Of course, this function  $d'$  is constant on conjugation classes, hence passes to the quotient  $m'_\Gamma(n)$ ; the restriction of this function to the space  $m_\Gamma^o(n)$  is the function  $d$ .

**Proof**

Suppose  $\rho_k \rightarrow \rho$ . By construction,

$$\{\rho \mid d'(\rho) < a\} = \left\{ \rho \mid \inf_{x \in \partial\mathbb{H}^n} \max_{s \in S} d(x, \rho(s)x) < a \right\} = \{\rho \mid \exists x \in \mathbb{H}^n, \forall s \in S, d(x, \rho(s)x) < a\}$$

is open (that is, the function  $d'$  is upper semi-continuous). For all  $\varepsilon$ , we thus have  $d'(\rho_k) < d'(\rho) + \varepsilon$ , for  $k$  large enough. In particular  $d'$  is continuous at every  $\rho$  such that  $d'(\rho) = 0$ .

If  $\rho$  has at least one global fixed point  $x \in \partial\mathbb{H}^n$ , then  $d'(\rho)$  is the maximum of translation distances of the  $\rho(s)$ ,  $s \in S$ , such that  $\rho(s)$  is loxodromic. Let  $s_0$  be the element of  $S$  maximizing this translation distance. Then, if  $\varepsilon > 0$ , for all  $k$  large enough,  $\rho_k(s_0)$  is loxodromic, and its translation distance is close to that of  $\rho(s_0)$ , so that  $d'(\rho_k) > d'(\rho) - \varepsilon$ .

Now suppose that  $\rho$  is non parabolic. Let  $\varepsilon > 0$ . Since the topology of  $Isom^+(\mathbb{H}^n)$  coincides with the compact-open topology and since the set

$$F_{3\varepsilon} = \left\{ x \in \mathbb{H}^n \mid \max_{s \in S} d(x, \rho(s)x) \leq 3\varepsilon \right\}$$

is compact, for all  $k$  large enough, for all  $x \in F$  and all  $s \in S$ , we thus have  $|d(x, \rho_k(s)x) - d(x, \rho(s)x)| < \varepsilon$ . Hence, the convex function  $x \mapsto \max_{s \in S} d(x, \rho_k(s)x)$  has a local minimum in the open set  $\min_{3\varepsilon}(\rho)$ . It follows that this minimum is global, whence  $|d(\rho) - d(\rho_k)| \leq 4\varepsilon$  for  $k$  large enough.  $\square$

We turn to a second step towards the proof of theorem 2.5.

**Proposition 2.12** *The space  $m_\Gamma^o(n)$ , equipped with the induced topology, is Hausdorff.*

**Proof**

Once again, the space  $m'_\Gamma(n)$  being locally second countable, we can use sequences in this space.

Let  $[\rho_1], [\rho_2] \in m_\Gamma^o(n)$ . Suppose that  $[\rho_1]$  and  $[\rho_2]$  cannot be separated by open sets. This means that there exists a sequence  $([\rho_k])_k \in (m'_\Gamma(n))^\mathbb{N}$  converging both to  $[\rho_1]$  and  $[\rho_2]$ , in other words, there exist  $g_k, h_k \in SO_\mathbb{R}^+(n, 1)$  such that  $g_k \rho_k g_k^{-1} \rightarrow \rho_1$  and  $h_k \rho_k h_k^{-1} \rightarrow \rho_2$ . Up to conjugating  $\rho_k$  by  $h_k$ , we may suppose that  $h_k = 1$ .

First suppose that  $\rho_2$  is non parabolic. Let  $x \in \min(\rho_1)$ ; fix  $\varepsilon > 0$ . Then for all  $\gamma \in \Gamma$ ,  $d(g_k \rho_k(\gamma) g_k^{-1} x, x) = d(\rho_k(\gamma) g_k^{-1} x, g_k^{-1} x)$ , hence for all  $k$  large enough,  $g_k^{-1} x \in \min_\varepsilon(\rho_2)$ , since  $d(\rho_1) = d(\rho_2)$ . Since  $\min_\varepsilon(\rho_2)$  is bounded,  $g_k$  stays in a compact set (by proposition 2.2) and hence has a converging subsequence. Thus, up to taking subsequences, we have  $g_k \rightarrow g_\infty \in SO_\mathbb{R}^+(n, 1)$ , and  $\rho_1$  and  $\rho_2$  are conjugate. Of course this argument still works after exchanging the roles of  $\rho_1$  and  $\rho_2$ .

Now suppose that  $\rho_2$  has at least two distinct fixed points in  $\partial\mathbb{H}^n$ . Fix  $x \in \min(\rho_1)$ . Then, as before, for all  $\varepsilon > 0$  and all  $k$  large enough we have  $g_k^{-1}(x) \in \min_\varepsilon(\rho_2)$ . If  $d(x, g_k^{-1}x)$  is bounded then we can finish as in the preceding case. Let us then suppose that up to considering a subsequence, the sequence  $(g_k^{-1}x)_k$  converges to a point  $r_1 \in \partial\mathbb{H}^n$ . Then  $r_1$  is a fixed point of  $\rho_2$ . Choose another fixed point  $r_2 \in \partial\mathbb{H}^n$  of  $\rho_2$ . Denote by  $r_k$  the second boundary point of the axis  $(r_2, g_k^{-1}x)$ . Then the sequence  $(r_k)_k$  in  $\partial\mathbb{H}^n$  converges to  $r_1$ . Then there exists a sequence  $(u_k)_{k \in \mathbb{N}}$  of elements of  $Isom^+(\mathbb{H}^n)$ , converging to  $id_{\mathbb{H}^n}$  and such that  $u_k$  fixes  $r_2$ , and sends  $r_k$  to  $r_1$ . Since  $r_k$  converges to  $r_1$ , which is distinct of  $r_2$ , the distance of  $x$  to the axis  $(r_2, r_k)$  is bounded: there exists  $B > 0$  such that for all  $k$ ,  $d(x, (r_2, r_k)) < B$ . Denote by  $p_k$  the projection of  $x$  on the axis  $(r_2, r_k)$ , and denote by  $\varphi_k$  the hyperbolic element of axis  $(r_1, r_2)$  which sends  $u_k g_k^{-1}x$  to  $u_k p_k$ . Then  $u_k^{-1} \varphi_k u_k$  sends  $g_k^{-1}x$  to  $p_k$ , so that  $d(x, g_k \cdot (u_k^{-1} \varphi_k u_k)^{-1} x) < B$ . Hence, by proposition 2.2, up to extract it, the sequence  $g_k (u_k^{-1} \varphi_k u_k)^{-1}$  converges to an element  $\varphi \in Isom^+(\mathbb{H}^n)$ . We have  $g_k \rho_k g_k^{-1} \rightarrow \rho_1$  thus  $(u_k^{-1} \varphi_k u_k) \rho_k (u_k^{-1} \varphi_k u_k)^{-1} \rightarrow \varphi^{-1} \rho_1 \varphi$ . Denote  $\rho'_k = u_k \rho_k u_k^{-1}$ . Since  $\rho_k$  converges to  $\rho_2$  and since  $u_k$  converges to  $id_{\mathbb{H}^n}$ , we have again  $\rho'_k \rightarrow \rho_2$ . Similarly, we also have  $\varphi_k \rho'_k \varphi_k^{-1} \rightarrow \varphi^{-1} \rho_1 \varphi$ . Up to conjugate everything, we may suppose that the axis  $(r_1, r_2)$  is the axis  $(0, \infty)$ . Since  $\rho_2$  and  $\varphi_k$  preserve that axis, for every  $\gamma \in \Gamma$  we can write the elements  $\rho_2(\gamma)$ ,  $\rho'_k(\gamma)$  and  $\varphi_k$  under the form

$$\rho_2(\gamma) = \begin{pmatrix} \lambda(\gamma) & 0 & (0) \\ 0 & \frac{1}{\lambda(\gamma)} & (0) \\ (0) & (0) & A(\gamma) \end{pmatrix}, \quad \rho'_k(\gamma) = \begin{pmatrix} a_k(\gamma) & b_k(\gamma) & X_k(\gamma) \\ c_k(\gamma) & d_k(\gamma) & Y_k(\gamma) \\ Z_k(\gamma) & W_k(\gamma) & A_k(\gamma) \end{pmatrix}$$



and  $\varphi_k = \begin{pmatrix} t_k & (0) \\ (0) & \frac{1}{t_k} I_{n-1} \end{pmatrix}$ , where  $\lim_{k \rightarrow +\infty} t_k = +\infty$ , up to conjugate in order to exchange the points 0 and  $\infty$  in  $\partial\mathbb{H}^n$ . Then

$$\varphi_k \rho'_k(\gamma) \varphi_k^{-1} = \begin{pmatrix} a_k(\gamma) & \frac{1}{t_k^2} b_k(\gamma) & \frac{1}{t_k} X_k(\gamma) \\ t_k^2 c_k(\gamma) & d_k(\gamma) & t_k Y_k(\gamma) \\ t_k Z_k(\gamma) & \frac{1}{t_k} W_k(\gamma) & A_k(\gamma) \end{pmatrix}$$

so that

$$\varphi \rho_1(\gamma) \varphi^{-1} = \begin{pmatrix} \lambda(\gamma) & 0 & (0) \\ r(\gamma) & \frac{1}{\lambda(\gamma)} & Y(\gamma) \\ Z(\gamma) & (0) & A(\gamma) \end{pmatrix}.$$

Since  $[\rho_1] \in m_\Gamma$ , the representation  $\varphi \rho_1 \varphi^{-1}$  fixes another point of  $\partial\mathbb{H}^n$  than 0; denote it  $r_3$ . There exists an isometry  $\phi \in Isom^+(\mathbb{H}^n)$  fixing 0 and sending  $r_3$  to  $\infty$ , and now  $\phi \varphi \rho_1 \varphi^{-1} \phi^{-1}$  and  $\rho_2$  are conjugate.  $\square$

Now we prove another step towards theorem 2.5:

**Proposition 2.13** *On the set  $m_\Gamma^o(n)$ , the induced topology defined by  $m_\Gamma^o(n) \hookrightarrow m'_\Gamma(n)$  and the quotient topology defined by  $\pi: m'_\Gamma(n) \twoheadrightarrow m_\Gamma^o(n)$  coincide (in particular,  $\pi$  is continuous).*

**Proof**

By definition of  $\pi$ , and by proposition 2.12, for all  $x \in m'_\Gamma(n)$ ,  $\pi(x)$  is the unique element of  $m_\Gamma^o(n)$  such that every neighbourhood of  $\pi(x)$  contains  $x$ . Let  $U$  be an open subset of  $m_\Gamma^o(n)$  for the quotient topology. Then  $\pi^{-1}(U)$  is open in  $m'_\Gamma(n)$ , hence the set  $m'_\Gamma(n) \cap \pi^{-1}(U)$  is open for the induced topology. But  $m'_\Gamma(n) \cap \pi^{-1}(U) = U$ , since  $\pi$  is the identity on  $m'_\Gamma(n)$ . Consequently, in order to prove the proposition it suffices to prove that  $\pi$  is continuous.

Recall that  $m'_\Gamma(n)$  is locally second countable. In order to prove the continuity of  $\pi: m'_\Gamma(n) \rightarrow m_\Gamma^o(n)$ , we can use a sequential criterium.

Suppose that  $\rho_k, \rho_\infty \in R_\Gamma(n)$  and  $\rho_k \rightarrow \rho_\infty$ . We want to prove that  $\pi([\rho(k)])$  converges to  $\pi([\rho_\infty])$ . The elements  $\rho_k$  have either zero, or at least one fixed point in  $\partial\mathbb{H}^n$ . Up to consider two distinct subsequences, we may suppose that this situation does not depend on  $k$ . If  $\rho_k$  is non parabolic, for all  $k$ , then  $\pi([\rho_k]) = [\rho_k] \rightarrow [\rho_\infty]$ , but every neighbourhood of  $\pi([\rho_\infty])$  contains  $[\rho_\infty]$ , hence by definition the sequence  $\pi([\rho_k])$  also converges to  $\pi([\rho_\infty])$ . Suppose finally that  $\rho_k$  has at least one fixed point in  $\partial\mathbb{H}^n$ , for all  $k \in \mathbb{N}$ . By lemma 2.7, the representation  $\rho_\infty$  has at least one fixed point in  $\partial\mathbb{H}^n$ , and as before,  $[\rho_k]$  also converges to  $\pi([\rho_\infty])$ . It then follows from lemma 2.8 that  $\pi([\rho_k])$  converges to  $\pi([\rho_\infty])$ .  $\square$

Now that we proved that these two topologies coincide, we will equip the set  $m_\Gamma^o(n)$  with this topology in the sequel, without having to precise which topology we consider. We shall finish the proof of theorem 2.5 now:

**Corollary 2.14** *The function  $d: m_\Gamma^o(n) \rightarrow \mathbb{R}_+$  is continuous, and proper. In particular, the space  $m_\Gamma^o(n)$  is locally compact.*

**Proof**

The map  $R_\Gamma(n) \rightarrow m_\Gamma^o(n)$  is open, and the continuous function  $d': R_\Gamma(n) \rightarrow \mathbb{R}_+$  is constant on the conjugacy classes, hence it defines a continuous function of  $m_\Gamma^o(n)$  in  $\mathbb{R}_+$ . Hence,  $d$  is continuous, by considering the induced topology on  $m_\Gamma^o(n)$ .

Now, let  $A > 0$ ; let us prove that  $d^{-1}([0, A])$  is compact. Fix  $x_0 \in \mathbb{H}^2$ , and denote by  $R_\Gamma^A \subset R_\Gamma(n)$  the set of representations  $\rho$  satisfying

$$\max_{s \in S} d(x_0, \rho(s)x_0) \leq A.$$

By proposition 2.2, this set  $R_\Gamma^A$  is compact. Denote here by  $p$  the map  $p: R_\Gamma(n) \rightarrow m_\Gamma^o(n)$ : it is continuous, and takes values in a Hausdorff space, hence  $p(R_\Gamma^A)$  is compact; it therefore suffices to check that  $p(R_\Gamma^A) = d^{-1}([0, A])$ .

Let  $[\rho] \in d^{-1}([0, A])$ . Then, by lemma 2.9, there exists a point  $x \in \mathbb{H}^n$  such that  $\max_{s \in S} d(x, \rho(s)x) = d(\rho)$ , and up to conjugating  $\rho$  we have  $x = x_0$ . We then have  $\rho \in R_\Gamma^A$ , so that  $[\rho] \in p(R_\Gamma^A)$ . Now let  $\rho \in R_\Gamma^A$ . By definition of  $d(\rho)$ , we have  $d(\rho) \leq A$ . And the map  $d$  passes to the quotient  $p: R_\Gamma(n) \rightarrow m_\Gamma^o(n)$ , hence  $d([\rho]) = d(\rho) \leq A$ , and  $p(\rho) \in d^{-1}([0, A])$ . This proves that  $d: m_\Gamma^o(n) \rightarrow \mathbb{R}_+$  is proper.

Since the space  $\mathbb{R}_+$  is locally compact, so is  $m_\Gamma^o(n)$ .  $\square$

The group  $Isom(\mathbb{H}^n)$  of isometries of  $\mathbb{H}^n$  which may not be orientation-preserving, acts on  $m_\Gamma^o(n)$  by conjugation. We denote by  $m_\Gamma^u(n)$  the quotient of  $m_\Gamma^o(n)$  by this action ( $u$  standing for “unoriented”).

In the case  $n = 2$ , we shall prove here that this quotient identifies to the space  $X_\Gamma(2)$  formed by characters of representations. For all  $\rho \in Hom(\Gamma, PSL(2, \mathbb{R}))$ , denote by  $\chi(\rho): \Gamma \rightarrow \mathbb{R}_+$  its character. Then:

**Proposition 2.15** *Let  $[\rho_1], [\rho_2] \in m_\Gamma^o(2)$ . Then  $\chi(\rho_1) = \chi(\rho_2)$  if and only if there exists an isometry  $u$  of  $\mathbb{H}^2$  such that  $\rho_1 = u\rho_2u^{-1}$ .*

**Proof**

Of course, the character is a conjugation invariant in  $Hom(\Gamma, PSL(2, \mathbb{R}))$ ; hence there is only one direction to prove. The proof given here inspires of the proof of proposition 1.5.2 of M. Culler and P. Shalen [CS83].

It can be checked very easily (see e.g. [Wol07], proposition 1.1.16) that a representation  $\rho$  is elementary (i.e., has a global fixed point in  $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ ) if and only if  $\text{Tr}(\rho([\gamma_1, \gamma_2]))$  for all  $\gamma_1, \gamma_2 \in \Gamma$ ; in particular, elementary representation are characterized by their characters.

Suppose first that  $\rho_1$  and  $\rho_2$  are not elementary, and suppose further that  $\rho_1$  is not “dihedral” (following e.g. [Pau89], we say that a representation is *dihedral* if it is not elementary but fixes globally a pair  $\{x, y\} \subset \partial\mathbb{H}^2$ ). Then there exists  $\gamma_0 \in \Gamma$  such that  $\rho_1(\gamma_0)$  is hyperbolic;  $\rho_2(\gamma_0)$  is hyperbolic

too, and up to conjugating  $\rho_1$  and  $\rho_2$  by an element of  $PSL(2, \mathbb{R})$  we have  $\rho_1(\gamma_0) = \rho_2(\gamma_0) = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ , with  $\lambda > 1$ . Now let  $\gamma_1 \in \Gamma$  such that  $\rho_1(\gamma_1)$  has no fixed points in common with  $\rho_1(\gamma_0)$ , and is not an elliptic element whose fixed point is on the axis of  $\rho_1(\gamma)$  (this is always possible, when  $\rho_1$  is not dihedral), and denote  $\rho_1(\gamma_1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $\rho_2(\gamma_1) = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ . Then for all  $n \in \mathbb{Z}$ ,  $|a\lambda^n + \frac{d}{\lambda^n}| = |a'\lambda^n + \frac{d'}{\lambda^n}|$ , hence, up to change the signs of  $a'$ ,  $b'$ ,  $c'$  and  $d'$  we have  $a = a'$  and  $d = d'$ . Up to conjugating by diagonal matrices we can also suppose that  $|b| = |b'| = 1$ , since  $\gamma_0$  and  $\gamma_1$  do not share any fixed points. Up to conjugate by the reflection of axis  $(0, \infty)$ , we can further suppose that  $b = b' = 1$ , and then  $c = c'$ .

Now consider any  $\gamma \in \Gamma$  and denote  $\rho_1(\gamma) = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$  and  $\rho_2(\gamma) = \begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix}$ . Then, for all  $\gamma_1$ , up to change the signs of  $x'$ ,  $y'$ ,  $z'$  and  $t'$  we have  $x = x'$  and  $t = t'$ . Then, the equality  $\text{Tr}(\rho_1(\gamma_0^n \gamma_1 \gamma)) = \text{Tr}(\rho_2(\gamma_0^n \gamma_1 \gamma))$  yields

$$\left| \lambda^n(ax + bz) + \frac{1}{\lambda^n}(cy + dt) \right| = \left| \lambda^n(ax + bz') + \frac{1}{\lambda^n}(cy' + dt) \right|$$

for all  $n \in \mathbb{N}$ , where  $z = z'$  and  $y = y'$  or  $z = -z' + \frac{2ax}{b}$  and  $y = -y' + \frac{2dt}{c}$ . We can check that this last case implies that  $\rho_1(\gamma_1)$  is an elliptic element whose fixed point is on the axis of  $\rho_1(\gamma_0)$ , and we have supposed that this is not the case. This finishes the proof, in the “generic” case. Of course,  $\rho_1$  and  $\rho_2$  play symmetric roles, hence this also covers the case where  $\rho_2$  is non dihedral.

If  $\rho_1$  and  $\rho_2$  are dihedral, then as before, we can find  $\gamma_0, \gamma_1 \in \Gamma$  such that  $\rho_1(\gamma_0) = \rho_2(\gamma_0) = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$  with  $\lambda > 1$ , and  $\rho_1(\gamma_1) = \rho_2(\gamma_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . For all  $\gamma \in \Gamma$ ,  $\rho_i(\gamma)$  is then entirely determined by the traces of  $\rho_i(\gamma_0^n \gamma)$  and  $\rho_i(\gamma_0^n \gamma_1 \gamma)$ .

We still have to deal with the case when  $\rho_1$  and  $\rho_2$  are elementary. Two distinct situations may occur. If  $\rho_1$  possesses at least one global fixed point in  $\partial\mathbb{H}^2$ , then it possesses two, hence all the elements  $\rho_1(\gamma)$  are hyperbolic and share the same axis;  $\rho_1$  then identifies to an action of  $\Gamma$  on an axis  $\mathbb{R}$  by translations, which is determined by its character. If  $\rho_1$  possesses a (unique) global fixed point in  $\mathbb{H}^2$ , then so does  $\rho_2$ . The representations  $\rho_1$  and  $\rho_2$  factor through morphisms  $\varphi_i: \Gamma \rightarrow \mathbb{R}$ , such that  $|\cos \varphi_1| = |\cos \varphi_2|$ . We can check again that  $\rho_1$  and  $\rho_2$  are conjugate by an isometry.  $\square$

The two quotients  $m_\Gamma^u(2)$  and  $X_\Gamma(2)$  of the space  $R_\Gamma(2)$  are therefore identical. Hence,  $m_\Gamma^o(2)$  is just an oriented version of the space of characters.

## 2.2 Recalls on the Bestvina-Paulin compactification of $m_g^u$

### 2.2.1 Equivariant Gromov topology

We are now going to recall F. Paulin's construction of the compactification of representation spaces, in order to adapt it to the whole space  $m_\Gamma^u(n)$ . We refer to [Pau04] for an efficient exposition of this construction.

Here we will be interested in metric spaces  $(X, d)$ , equipped with actions of  $\Gamma$  by isometries, i.e., morphisms  $\rho: \Gamma \rightarrow \text{Isom}(X, d)$ ; we say that  $(\rho, X, d)$  is equivalent to  $(\rho', X', d')$  if there exists a  $\Gamma$ -equivariant isometry between  $X$  and  $X'$ . Let  $\mathcal{E}$  be a set of classes of equivariant isometry of actions of  $\Gamma$  on metric spaces. If  $(\rho, X, d) \in \mathcal{E}$  (in order to avoid too heavy notations, we denote again by  $(\rho, X, d)$  its class of equivariant isometry), if  $\varepsilon > 0$ , if  $\{x_1, \dots, x_p\}$  is a finite collection in  $X$  (denote it  $K$ ) and if  $P$  is a finite subset of  $\Gamma$ , we define  $U_{K, \varepsilon, P}(\rho, X, d)$  as the subset of  $\mathcal{E}$  consisting of those  $(\rho', X', d')$  such that there exist  $\{x'_1, \dots, x'_p\} \in X'$ , such that for all  $g, h \in P$ , and all  $i, j \in \{1, \dots, p\}$ , we have

$$|d(\rho(g) \cdot x_i, \rho(h) \cdot x_j) - d'(\rho'(g) \cdot x'_i, \rho'(h) \cdot x'_j)| < \varepsilon.$$

In that case we also say that  $\{x'_1, \dots, x'_p\}$  is a  $P$  equivariant  $\varepsilon$ -approximation of the collection  $K$ . The sets  $U_{K, \varepsilon, P}(\rho, X, d)$  form a basis of open sets of a topology, called the *equivariant Gromov topology* (see [Pau88, Gui98]).

By definition, every representation  $\rho \in R_\Gamma(n)$  defines an action of  $\Gamma$  by isometries on the metric space  $(\mathbb{H}^n, d_{\mathbb{H}^n})$  (where  $d_{\mathbb{H}^n}$  is the usual distance on  $\mathbb{H}^n$ ), and every conjugation by an isometry of  $\mathbb{H}^n$  defines an equivariant isometry. Hence, every element  $[\rho] \in m_\Gamma^u(n)$  defines a unique equivariant isometry class of actions  $(\rho, \mathbb{H}^n, d_{\mathbb{H}^n})$ ; we can therefore consider the set  $m_\Gamma^u(n)$  as a set of (equivariant isometry classes of) actions of  $\Gamma$  on  $(\mathbb{H}^n, d_{\mathbb{H}^n})$  and we can equip this set with the equivariant Gromov topology. Then:

**Proposition 2.16 (F. Paulin [Pau88], proposition 6.2)** *On the set  $m_\Gamma^u(n)$ , the usual topology and the equivariant Gromov topology coincide.*

Since we will have to adapt this proposition to the oriented case when  $n = 2$ , we recall here the proof given in [Pau88], in that case. Note, anyway, that the general case  $n \geq 2$  is proved similarly.

**Proof** in the case  $n = 2$ .

The usual topology is of course finer than the equivariant Gromov topology, since the distances considered are continuous for the usual topology on  $m_\Gamma^u$ .

Conversely, fix  $\varepsilon > 0$ , and let  $[\rho_k]$  be a sequence converging to  $[\rho_\infty]$  for the equivariant Gromov topology. Let us prove that up to conjugate these representations  $\rho_k \rightarrow \rho_\infty$  (since  $\mathbb{H}^2$  is separable, the space the space  $m_\Gamma^u(n)$  is locally second countable; hence we can indeed use sequences in that space). Consider three points  $x_1, x_2, x_3 \in \mathbb{H}^2$  which form a non degenerate triangle. Then, for all  $\varepsilon' > 0$ , and for  $k$  large enough, there exists a collection  $\{x_1^k, x_2^k, x_3^k\} \in \mathbb{H}^2$  such that for every  $i, j \in \{1, 2, 3\}$  and every  $s_1, s_2 \in S$ ,

$$|d(\rho_\infty(s_1)x_i, \rho_\infty(s_2)x_j) - d(\rho_k(s_1)x_i^k, \rho_k(s_2)x_j^k)| < \varepsilon.$$

Now let  $S = \{s_1, \dots, s_n\}$  and  $y_1 = \rho_\infty(s_1)x_1, y_2 = \rho_\infty(s_1)x_2, \dots, y_{3n} = \rho_\infty(s_n)x_3$ , and similarly define  $y_1^k, \dots, y_{3n}^k \in \mathbb{H}^2$ . The following fact will enable us to conclude:

**Fact 2.17** *For all  $\varepsilon > 0$ , there exists  $\varepsilon' > 0$  such that for all  $x_1, \dots, x_n, x'_1, \dots, x'_n$  in  $\mathbb{H}^2$ , if for all  $i, j$ ,  $|d(x_i, x_j) - d(x'_i, x'_j)| < \varepsilon'$  then there exists an isometry  $\varphi$  of  $\mathbb{H}^2$  such that  $d(\varphi(x_i), x'_i) < \varepsilon$ .*

This is left as an exercise (see e.g. [Wol07], proposition 1.1.8), and follows from the fact that the sine and cosine laws, in the hyperbolic plane, can be used to recover continuously a triangle from its three lengths. In particular, up to conjugate  $\rho_k$  by an isometry of  $\mathbb{H}^2$ , we thus have  $d(y_i, y'_i) < \varepsilon$ , and for every  $s \in S$ , it follows that  $d(x_i, \rho_k(s) \cdot \rho_\infty^{-1}(s)x_i) < \varepsilon$ , for the three non aligned points  $x_1, x_2, x_3$ , so that  $\rho_k \rightarrow \rho_\infty$  in the usual topology.  $\square$

For all  $\rho \in m_\Gamma^u(n)$ , denote

$$\ell(\rho) = \max(1, d(\rho))$$

and equip the set  $\mathbb{H}^n$  with the distance  $\frac{d_{\mathbb{H}^n}}{\ell(\rho)}$ . From now on, every element  $[\rho] \in m_\Gamma^u(n)$  will be associated to  $(\rho, \mathbb{H}^n, \frac{d_{\mathbb{H}^n}}{\ell(\rho)})$  instead of  $(\rho, \mathbb{H}^n, d_{\mathbb{H}^n})$  (this is another realization of  $m_\Gamma^u(n)$  as a set of (classes of) actions of  $\Gamma$  on  $\mathbb{H}^n$ , and the equivariant Gromov topology is still the same, by proposition 2.16 and corollary 2.14).

As such, the equivariant Gromov topology does not separate any action on a space from the restricted actions on invariant subspaces. In particular, if we consider the whole space  $m_\Gamma^u(n)$  as well as actions of  $\Gamma$  on  $\mathbb{R}$ -trees, including actions on lines, this will yield a non-Hausdorff space, since some of the elements of  $m_\Gamma^u(n)$  have an invariant line. In order to get rid of this little degeneracy, we are going to modify slightly the definition of the equivariant Gromov topology, so that elementary representations in  $m_\Gamma^u$  will be separated from the corresponding actions on lines, when considered as actions on  $\mathbb{R}$ -trees. If  $\mathcal{E}$  is a set of (classes of) actions of  $\Gamma$  by isometries on spaces which are *hyperbolic in the sense of Gromov*, put  $U'_{K, \varepsilon, P}(\rho, X, d)$  to be the subset of  $\mathcal{E}$  consisting of those  $(\rho', X', d')$  such that there exist  $\{x'_1, \dots, x'_p\} \in X'$ , such that for all  $g, h \in P$ , and all  $i, j \in \{1, \dots, p\}$ , we have

$$|d(\rho(g) \cdot x_i, \rho(h) \cdot x_j) - d'(\rho'(g) \cdot x'_i, \rho'(h) \cdot x'_j)| < \varepsilon \text{ and } |\delta(X) - \delta(X')| < \varepsilon.$$

As we shall see very soon (proposition 2.18), this extra condition changes the equivariant Gromov topology only at the neighbourhood of elementary representations. It is comparable to the fact of adding 2 to characters, as it is done by J. Morgan and P. Shalen in [MS84], in order not to bother with the neighbourhood of the trivial representation.

### 2.2.2 The space $\overline{m_\Gamma^u(n)}$

Let us begin this section by a few recalls. Let  $X$  be a topological space. A *compactification* of  $X$  is a couple  $(\overline{X}, i)$  such that  $\overline{X}$  is a compact Hausdorff space,  $i: X \hookrightarrow \overline{X}$  is a homeomorphism on its image, and such that  $i(X)$  is open and dense in  $\overline{X}$ .

Note that we request  $\overline{X}$  to be Hausdorff. In particular, a space  $X$  needs to be locally compact in order to admit a compactification. If  $X$  is a locally compact space and  $(\overline{X}, i)$  is a compactification of  $X$ , the compact set  $\overline{X} \setminus X$  is called the *boundary* of  $X$ , and is denoted  $\partial\overline{X}$ , or  $\partial X$  if no confusion is possible between different compactifications of  $X$ . The points in the boundary are called the *ideal points* of the compactification.

Note that, if  $X$  is locally compact and if  $(\overline{X}, i)$  is a compactification of  $X$ , then the open subsets of  $\overline{X}$  containing  $\partial\overline{X}$  are precisely the complements, in  $\overline{X}$ , of the compact subsets of  $X$  (indeed, if  $K \subset X$  is compact then  $i(K)$  is compact, hence  $\overline{X} \setminus i(K)$  is open and contains  $\partial\overline{X}$ , and if  $U$  is an open subset of  $\overline{X}$  containing  $\partial\overline{X}$  then  $\overline{X} \setminus U$  is compact, contained in  $i(X)$ , so that it is the image of a compact set, since  $i$  is a homeomorphism on its image).

Finally, if  $X$  is a locally compact space, if  $Y$  is Hausdorff and  $f: X \rightarrow Y$  is continuous and has a relatively compact image, then we can define a compactification of  $X$  as follows (see [MS84], p. 415). Denote by  $\widehat{X} = X \cup \{\infty\}$  the Alexandrov compactification of  $X$  (that is, the one-point compactification), we define  $i: X \rightarrow \widehat{X} \times Y$  by  $i(x) = (x, f(x))$ . Denote  $\overline{X}$  the closure of  $i(X)$  in  $\widehat{X} \times Y$ . Then  $(\overline{X}, i)$  is a compactification of  $X$ ; we say that it is the *compactification defined by  $f$* .

We say that an action of  $\Gamma$  on an  $\mathbb{R}$ -tree  $T$  is *minimal* if  $T$  has no proper invariant sub-tree. The equivalence classes of  $\mathbb{R}$ -trees equipped with minimal actions of  $\Gamma$  by isometries, up to equivariant isometry, form a set, and we denote  $\mathcal{T}'(\Gamma)$  the subset formed by trees not reduced to a point. In order to exhibit this set, one can prove that the  $\mathbb{R}$ -tree  $T$  and the action of  $\Gamma$  are entirely determined by the set  $\{d(p, \gamma \cdot p) | \gamma \in \Gamma\}$  (see [AM85, MS84]).

We have modified the definition of the equivariant Gromov topology so that we would be considering Hausdorff spaces, and for this reason we are also going to restrict the set of  $\mathbb{R}$ -trees we consider. If  $(\rho, T)$  possesses an end which is globally fixed by  $\rho$  (we then say that this action is *reducible*, see e.g. [Pau89]), then  $(\rho, T, d)$  is not separated, in the equivariant Gromov topology, from the action on a line which has the same translation lengths. Therefore, we shall restrict ourselves to the subset  $\mathcal{T}(\Gamma) \subset \mathcal{T}'(\Gamma)$  consisting of actions on lines and actions on  $\mathbb{R}$  trees without fixed ends, such that  $\min_{x_0 \in T} \max_{\gamma \in S} d(x_0, \gamma \cdot x_0) = 1$ .

In the sequel, we equip the set  $\overline{m_\Gamma^u(n) \cup \mathcal{T}(\Gamma)}$  with the modified equivariant Gromov topology, and we denote by  $\overline{m_\Gamma^u(n)}$  the closure of the set  $m_\Gamma^u(n)$  in the space  $m_\Gamma^u(n) \cup \mathcal{T}(\Gamma)$ .

The following proposition is simply a detailed version of an argument of F. Paulin, in [Pau88], saying that  $\mathbb{R}$ -trees and hyperbolic structures are not undistinguishable by the equivariant Gromov topology, so that the modified

equivariant Gromov topology changes it only at the boundary of degenerate representations.

We say that a finite collection  $K$  is *big* if it contains four points  $A, B_1, B_2, B_3$  such that  $d(A, B_i) = 1$ ,  $d(B_1, B_3) = 2$  and  $d(B_1, B_2) = d(B_2, B_3)$ . In an  $\mathbb{R}$ -tree, this means that their convex hull is a tripod of centre  $A$ , and this implies that  $d(B_1, B_2) = 2$ . In  $\mathbb{H}^n$ , this means that the segments  $[B_1, B_3]$  and  $[B_2, A]$  meet orthogonally at  $A$ .

**Proposition 2.18** *Let  $(\rho, X, d) \in m_\Gamma^u(n) \cup \mathcal{T}(\Gamma)$  and let  $K \subset X$  be a big finite collection. Let  $\varepsilon > 0$ . Then for all  $\varepsilon' > 0$  small enough, for all  $(\rho', X', d') \in U_{K, \varepsilon', \{1\}}(\rho, X, d)$ , we have  $|\delta(X') - \delta(X)| < \varepsilon$ .*

Here, by  $\varepsilon'$  small enough, we mean:  $\varepsilon' < v(\delta(X), \varepsilon)$ , where  $v: \mathbb{R}_+ \times \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$  is some (universal) function.

**Proof**

Let  $\varepsilon' > 0$ , and let  $K' = \{A', B'_1, B'_2, B'_3\} \subset X'$  be an  $\varepsilon'$ -approximation of  $K$ . Put  $B''_1 = B'_1$ . We have  $|d(B'_1, B'_3) - 2| < \varepsilon'$  so we can choose a point  $B''_3 \in X'$  such that  $d(B'_3, B''_3) < \varepsilon'$  and  $d(B'_1, B''_3) = 2$ . Denote by  $r(t)$  the geodesic segment, in  $X'$ , such that  $r(0) = B''_1$  and  $r(2) = B''_3$ , and put  $A'' = r(1)$ . Then the  $CAT(0)$  property on  $X'$  implies that  $d(A', A'') < \sqrt{2\varepsilon' + \varepsilon'^2}$ . Finally, we have  $|d(A', B'_2) - 1| < \varepsilon'$  so there exists  $B''_2 \in X$  such that  $d(B'_2, B''_2) < \varepsilon' + \sqrt{2\varepsilon' + \varepsilon'^2}$  and  $d(A'', B''_2) = 1$ . We also have the inequalities  $|d(B'_1, B''_2) - 1| < 2\varepsilon' + \sqrt{2\varepsilon' + \varepsilon'^2}$ , and  $|d(B'_3, B''_2) - d(B''_1, B''_2)| < 5\varepsilon' + 2\sqrt{2\varepsilon' + \varepsilon'^2}$ .

First suppose that  $\delta(X) \neq 0$ , and denote  $x = \frac{\delta_{\mathbb{H}^2}}{\delta(X)}$ . The cosine law I, in  $(\mathbb{H}^2, d_{\mathbb{H}^2})$ , implies that  $\cosh(xd(B_1, B_2)) = \cosh^2(x)$ . For all  $\varepsilon'$  small enough, we have  $d(A'', B''_i) = 1$ ,  $d(B''_1, B''_3) = 2$ ,  $d(B''_1, B''_2) < 2$  and  $d(B''_3, B''_2) < 2$ , so that  $X'$  cannot be an  $\mathbb{R}$ -tree. Put also  $x' = \frac{\delta_{\mathbb{H}^2}}{\delta(X')}$ . Then the cosine law I in  $X'$  implies that  $\cosh(x'd(B''_1, B''_2)) + \cosh(x'd(B''_2, B''_3)) = 2\cosh^2(x')$ . It follows from the study of the function  $F: [0, 2] \times \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by  $F(b, x) = \cosh^2(x) - \cosh(bx)$ , that by taking  $\varepsilon'$  small enough we can force  $x$  and  $x'$  to be arbitrarily close.

Now suppose that  $\delta(X) = 0$ . If  $\delta(X') = 0$  then there is nothing to do. Otherwise, we have again  $\cosh(x'd(B''_1, B''_2)) + \cosh(x'd(B''_2, B''_3)) = 2\cosh^2(x')$ , where, for  $\varepsilon'$  small enough, the distances  $d(B''_1, B''_2)$  and  $d(B''_2, B''_3)$  can be taken arbitrarily close to 2, which implies that  $x'$  can be forced to be arbitrarily large.  $\square$

In particular, the modified equivariant Gromov topology and the equivariant Gromov topology coincide in  $m_\Gamma^u(n)$ .

Now, every arguments of M. Bestvina and F. Paulin (see [Bes88, Pau88, Pau04]) work, and we have the following.

**Theorem 2.19 (M. Bestvina, F. Paulin)** *The space  $\overline{m_\Gamma^u(n)}$ , equipped with the function  $m_\Gamma^u(n) \hookrightarrow \overline{m_\Gamma^u(n)}$ , is a natural compactification of  $m_\Gamma^u(n)$ .*

By “natural”, we mean that the action of  $Out(\Gamma)$  on  $m_\Gamma^u(n)$  extends continuously to an action of  $Out(\Gamma)$  on  $\overline{m_\Gamma^u(n)}$ .

We refer to [Pau04, Pau89] and [KL95] for a complete proof of this result.

### 2.2.3 Other compactifications

We now give a (very) short recall on the compactification of  $X_{\Gamma, SL(2, \mathbb{R})}$  by J. Morgan and P. Shalen. The numerable collection  $(f_\gamma)_{\gamma \in \Gamma}$ ,  $f_\gamma: \chi \mapsto \chi(\gamma)$  generates the coordinate ring of  $X_{\Gamma, SL(2, \mathbb{R})}$ . Denote  $P\mathbb{R}^\Gamma$  the quotient of  $[0, +\infty)^\Gamma \setminus \{0\}$  by positive multiplication, and let  $\theta: X_{\Gamma, SL(2, \mathbb{R})} \rightarrow P\mathbb{R}^\Gamma$  defined by  $\theta(x) = [\log(|f_\gamma(x)| + 2)]_{\gamma \in \Gamma}$ . J. Morgan and P. Shalen proved (see [MS84], proposition I.3.1) that the image of  $X_{\Gamma, SL(2, \mathbb{R})}$  under  $\theta$  is relatively compact, so that  $\theta$  defines a compactification of  $X_{\Gamma, SL(2, \mathbb{R})}$ .

We now restrict to the group  $\Gamma = \pi_1 \Sigma_g$  with  $g \geq 2$ , and we denote by  $m_g^u$  the space  $m_\Gamma^u(2)$ . Then the absolute value of the Euler class is defined on  $m_g^u$ , and we denote  $m_{g, \text{even}}^u$  the subspace of  $m_g^u$  consisting of representations of even Euler class (recall, indeed, that a representation  $\rho: \pi_1 \Sigma_g \rightarrow PSL(2, \mathbb{R})$  lifts to  $SL(2, \mathbb{R})$  if and only if its Euler class is even; we will see that in section 2.3.5). Then the map  $\theta$  factors through  $\theta': m_{g, \text{even}}^u \rightarrow P\mathbb{R}^\Gamma$ , which defines a compactification  $\overline{m_{g, \text{even}}^u}^{MS}$  of  $m_{g, \text{even}}^u$ . Similarly, the functions  $m_{g, \text{even}}^u \hookrightarrow m_g^u \hookrightarrow \overline{m_g^u}$  define a compactification  $\overline{m_{g, \text{even}}^u}$  of that space. The ideal points of the compactifications  $\overline{m_{g, \text{even}}^u}$  and  $\overline{m_{g, \text{even}}^u}^{MS}$  are actions of  $\pi_1 \Sigma_g$  on  $\mathbb{R}$ -trees. These actions on  $\mathbb{R}$ -trees are irreducible, i.e., without global fixed points, or are actions on a line. The topology on  $\partial \overline{m_{g, \text{even}}^u}^{MS}$  is the *axis topology*; it is the coarsest topology such that the functions  $\ell_T(\gamma) = \inf_{x \in T} d(x, \gamma x)$  are continuous.

By the main theorem of [Pau89], the spaces  $\partial \overline{m_{g, \text{even}}^u}$  and  $\partial \overline{m_{g, \text{even}}^u}^{MS}$  are homeomorphic, and hence, as F. Paulin explains it in [Pau88], the spaces  $\overline{m_{g, \text{even}}^u}$  and  $\overline{m_{g, \text{even}}^u}^{MS}$  are homeomorphic (we refer to [MS84, Pau88, Pau89] for details). In particular, the corollary 1.4 concerns simply the space  $\overline{m_{g, \text{even}}^u}$  and it is under that form that we shall prove it in section 3.

## 2.3 Euler class

We shall now introduce the Euler class, for actions of groups on cyclically ordered sets. For the sake of completeness we shall give a complete construction, but equivalent definitions can be found in [Ghy87b, Thu]. We refer to [Cal04] for a detailed overview.

### 2.3.1 Cyclically ordered sets

**Definition 2.20** *Let  $X$  be a set. A (total) cyclic order on  $X$  is a function  $o: X^3 \rightarrow \{-1, 0, 1\}$  such that:*

- (i)  $o(x, y, z) = 0$  if and only if  $\text{card}\{x, y, z\} \leq 2$ ;
- (ii) For all  $x, y$  and  $z$ ,  $o(x, y, z) = o(y, z, x) = -o(x, z, y)$ ;



(iii) For all  $x, y, z$  and  $t$ , if  $o(x, y, z) = 1$  and  $o(x, z, t) = 1$  then  $o(x, y, t) = 1$ .

**Remark 2.21** If  $o(x, y, z) = 1$  and  $o(x, z, t) = 1$  then we also have  $o(x, z, t) = 1$  and  $o(y, z, t) = 1$ . Indeed,  $o(z, x, y) = o(z, t, x) = 1$  so, by condition (iii) of the definition,  $o(z, t, y) = 1$ , that is,  $o(y, z, t) = 1$ . Similarly,  $o(x, y, t) = 1$  so  $o(t, x, y) = 1$ , which, together with  $o(t, y, z) = 1$ , yields  $o(t, x, z) = 1$ , i.e.  $o(x, z, t) = 1$ . In other words, the transitivity relation (iii) implies all the other “natural” transitivity relations. In particular, for instance, on a set of 4 elements, there are as many total cyclic orders as injections of that set in the oriented circle, up to homeomorphism, that is, 6.

**Remark 2.22** In all this text, we use only total cyclic orders (every triple defines an order). Consequently, we will sometimes forget the word “total” when we refer to a cyclic order.

Now fix a set  $X$  equipped with a cyclic order  $o$  and a base point  $x_0 \in X$ .

**Definition 2.23** We set  $y <_{x_0} z$  if  $o(x_0, y, z) = 1$ , and  $y \leq_{x_0} z$  if  $y <_{x_0} z$  or  $y = z$ .

The following proposition follows directly from the properties of  $o$ .

**Proposition 2.24** The relation  $\leq_{x_0}$  is a total order on  $X \setminus \{x_0\}$ .

**Remark 2.25** Reciprocally, if  $\leq$  is a total order on  $X \setminus \{x_0\}$ , then there exists a unique cyclic total order on  $X$  which verifies  $o(x, y, z) = 1$  for all  $x, y, z \neq x_0$  such that  $x < y < z$ , and verifying  $o(x_0, y, z) = 1$  as soon as  $y < z$  (by enumerating the different possible cases we can check that the axioms of definition 2.20 are satisfied). For all  $x_0 \in X$ , these two constructions realize a bijection, and its inverse, between the set of total cyclic orders on  $X$  and the set of total orders on  $X \setminus \{x_0\}$ . In particular, on any set  $X$  there exists at least one total cyclic order, by the axiom of choice.

**Definition 2.26** On the set  $\mathbb{Z} \times X$  we put:

- $(m, x) <_{x_0} (n, y)$  when  $m < n$ , for any  $x, y \in X$ ,
- $(k, y) <_{x_0} (k, z)$  when  $y <_{x_0} z$ ,
- $(k, x_0) <_{x_0} (k, y)$  for all  $y \in X \setminus \{x_0\}$ ,

and we put  $(m, y) \leq_{x_0} (n, z)$  if  $(m, y) <_{x_0} (n, z)$  or  $(m, y) = (n, z)$ .

In particular, when restricted to the set  $\mathbb{Z} \times (X \setminus \{x_0\})$  it is the lexicographic order.

We check easily the following.

**Proposition 2.27** The relation  $\leq_{x_0}$  is a total order on the set  $\mathbb{Z} \times X$ .

**Example**

If  $X = \mathbb{S}^1$  and  $x_0 \in \mathbb{S}^1$ , then  $X \setminus \{x_0\}$  is an interval,

$$\begin{array}{c}
 \dots \\
 \{1\} \times (X \setminus \{x_0\}) \left( \begin{array}{c} \bullet(2, x_0) \\ \bullet(1, x_0) \end{array} \right) \\
 \{0\} \times (X \setminus \{x_0\}) \left( \begin{array}{c} \bullet(0, x_0) \end{array} \right) \\
 \{-1\} \times (X \setminus \{x_0\}) \left( \begin{array}{c} \bullet(-1, x_0) \end{array} \right) \\
 \dots
 \end{array}$$

and  $\mathbb{Z} \times X$  identifies naturally to  $\mathbb{R}$ , and this identification depends canonically on  $x_0$ .

Another example is the deck of cards. It is equipped with a cyclic order, which is preserved as we make a “cut”. The choice of the “cut” consists in choosing a card  $x_0$ , and determines a total order in the deck. Here our definition of the order on  $\mathbb{Z} \times X$  consists in choosing a cut, and then to put  $\mathbb{Z}$  copies of the deck the ones above the others.

### 2.3.2 Applications and lifts

We consider here a set  $X$  equipped with a total cyclic order  $o$ . We shall see that the bijections of  $X$  preserving  $o$  can be lifted to  $\mathbb{Z} \times X$  in some natural way (as soon as we take a base point  $x_0$ ), similarly as the orientation-preserving homeomorphisms of  $\mathbb{S}^1$  can be lifted naturally to its cover  $\mathbb{R}$ .

**Definition 2.28** *Let  $f : X \rightarrow X$  be any map. We call an arbitrarily lift of  $f$  any function  $\tilde{f}$  such that the following diagram commutes:*

$$\begin{array}{ccc}
 \mathbb{Z} \times X & \xrightarrow{\tilde{f}} & \mathbb{Z} \times X \\
 \downarrow pr_2 & & \downarrow pr_2 \\
 X & \xrightarrow{f} & X .
 \end{array}$$

*In that case we say that  $\tilde{f}$  projects on  $f$ .*

We define an application  $h : \mathbb{Z} \times X \rightarrow \mathbb{Z} \times X$  by  $h(n, x) = (n + 1, x)$ .

**Proposition 2.29** *Let  $f : \mathbb{Z} \times X \rightarrow \mathbb{Z} \times X$  be a bijection preserving the order  $\leq_{x_0}$  and which projects to  $id_X$ . Then there exists an integer  $n$  such that  $f = h^n$ .*

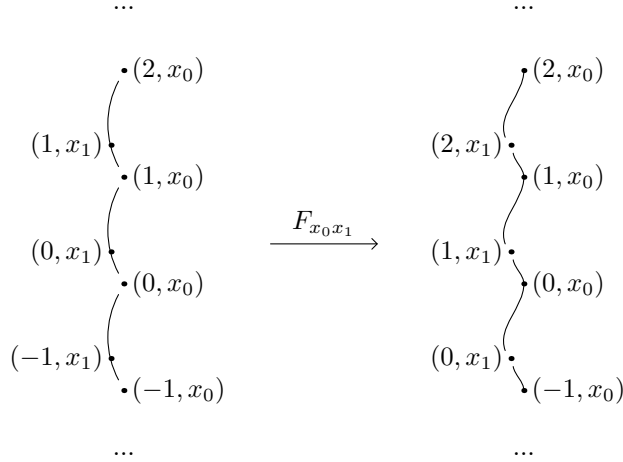
**Proof**

Let  $f$  be such a function; put  $n = pr_1(f(0, x_0))$ . Then the function  $pr_1 \circ f(\cdot, x_0) : \mathbb{Z} \rightarrow \mathbb{Z}$  is a bijection preserving the order, hence for all  $k \in \mathbb{Z}$ ,  $f(k, x_0) = (n + k, x_0)$ . Now, for all  $y \in X \setminus \{x_0\}$ , we have  $f(k, x_0) \leq_{x_0} f(k, y) \leq_{x_0} f(k + 1, x_0)$ , that is,  $(n + k, x_0) \leq_{x_0} f(k, y) \leq_{x_0} (n + k + 1, x_0)$ , hence  $f(k, y) = (n + k, y)$ , since  $f$  projects on  $id_X$ .  $\square$

**Proposition 2.30** *For all  $x_0, x_1 \in X$ , there exists a unique bijection  $F_{x_0x_1}$  of  $\mathbb{Z} \times X$  which projects on  $id_X$  such that for all  $a, b \in \mathbb{Z} \times X$ ,  $a <_{x_0} b \Leftrightarrow F_{x_0x_1}(a) <_{x_1} F_{x_0x_1}(b)$ , and such that  $F_{x_0x_1}(0, x_0) = (0, x_0)$ .*

**Proof**

If  $x_0 = x_1$ , then by the preceding proposition, the unique possible function is  $F_{x_0x_1} = id_{\mathbb{Z} \times X}$ . Now suppose that  $x_0 \neq x_1$ . The application  $F_{x_0x_1}$  must project on  $id_X$ , so that we can only change indices, in the way suggested by the following picture.



We put  $F_{x_0x_1}(k, x_0) = (k, x_0)$  and  $F_{x_0x_1}(k, x_1) = (k + 1, x_1)$ . For all  $y \in X \setminus \{x_0, x_1\}$ , if  $o(x_0, x_1, y) = -1$  we put  $F_{x_0x_1}(k, y) = (k, y)$  otherwise we put  $F_{x_0x_1}(k, y) = (k + 1, y)$ : we then have  $a <_{x_0} b \Leftrightarrow F_{x_0x_1}(a) <_{x_1} F_{x_0x_1}(b)$ . Now, if  $f$  is another bijection satisfying the same conditions, then  $F_{x_0x_1} \circ f^{-1}$  is a bijection of  $\mathbb{Z} \times X$  which preserves the order  $\leq_{x_0}$ , which projects on  $id_X$  and which fixes  $(0, x_0)$  hence, by proposition 2.29, it is the identity.  $\square$

Note that for all distinct points  $x_0, x_1 \in X$  we have  $F_{x_0x_1} \circ F_{x_1x_0} = h$ , contrarily to what our notation may suggest.

**Proposition 2.31** *Let  $X$  be a set equipped with a (total) cyclic order  $o$ , let  $x_0 \in X$ , and let  $f : X \rightarrow X$  be a bijection preserving  $o$ . Then  $f$  admits at least a lift  $\tilde{f}$  preserving the order  $\leq_{x_0}$ . Moreover, if  $\tilde{f}$  and  $\tilde{f}'$  are two such lifts then there exists  $n \in \mathbb{Z}$  such that  $\tilde{f}' = h^n \cdot \tilde{f}$ .*

**Proof**

Let  $f: X \rightarrow X$  preserving  $o$ . We check easily that the map  $F: \mathbb{Z} \times X \rightarrow \mathbb{Z} \times X$  defined by  $F(n, x) = (n, f(x))$  verifies:  $\forall a, b \in \mathbb{Z} \times X, a \leq_{x_0} b \Leftrightarrow F(a) \leq_{f(x_0)} F(b)$ . We therefore put  $\tilde{f} = F_{f(x_0)x_0} \circ f$ . Then  $\tilde{f}$  is indeed a lift of  $f$ , preserving  $\leq_{x_0}$ .

Now, if  $\tilde{f}'$  is another lift of  $f$  preserving  $\leq_{x_0}$ , then  $\tilde{f}' \circ \tilde{f}^{-1}$  is a lift of  $id_X$  preserving  $\leq_{x_0}$ , hence, by proposition 2.29,  $\tilde{f}' \tilde{f}^{-1} = h^n$ , for some  $n \in \mathbb{Z}$ .  $\square$

Now denote by  $\widetilde{Ord(X, o)}$  the group of bijections of  $X$  which preserve  $o$ , and by  $\widetilde{Ord(X, o, x_0)}$  the group formed by lifts of elements of  $\widetilde{Ord(X, o)}$  à  $\mathbb{Z} \times X$  which preserve the order  $\leq_{x_0}$ . By proposition 2.30, the conjugation by  $F_{x_0x_1}$  realizes a canonical isomorphism between the groups  $\widetilde{Ord(X, o, x_0)}$  and  $\widetilde{Ord(X, o, x_1)}$ , for all  $x_0, x_1 \in X$ . This group, considered up to isomorphism, is denoted  $\widetilde{Ord(X, o)}$ , and if there is no confusion possible about  $o$ , these two groups are denoted  $\widetilde{Ord(X)}$  and  $Ord(X)$ .

### 2.3.3 Euler class

Let  $(X, o)$  be a cyclically ordered set, let  $\Sigma$  be a (connected) surface, and let  $\rho: \pi_1 \Sigma \rightarrow \widetilde{Ord(X)}$  be a representation: we shall define an element  $e(\rho) \in H^2(\Sigma, \mathbb{Z})$ , depending (only) on  $\rho$ , and which we call the Euler class of the representation.

Since we are considering the representations of  $\pi_1 \Sigma$  in a non abelian group, we first need to choose a base point  $* \in \Sigma$ . Choose also a base point  $x_0 \in X$ . Let then  $C = C_0 \cup C_1 \cup C_2$ ;  $C_i = \{\sigma_\alpha^i\}_\alpha$ , be a cellulation of  $\Sigma$ , where every cell is equipped with an orientation, and such that  $* \in C_0$ . For each loop  $\gamma$  (based at  $*$ ) in the 1-skeleton, we can write  $\gamma$  as a word  $(\sigma_1^1)^{s_1} \cdots (\sigma_k^1)^{s_k}$ ,  $s_j = \pm 1$ , in the elements of  $C_1$ . We shall say that a function  $f: C_1 \rightarrow \widetilde{Ord(X)}$  projects on  $\rho$  if for every loop  $\gamma$  based at  $*$  in the 1-skeleton, the element  $(f(\sigma_1^1))^{s_1} \circ \cdots \circ (f(\sigma_k^1))^{s_k}$  (keeping the same notations for  $\gamma$ ), which we call  $f(\gamma)$ , is a lift of  $\rho(\gamma)$ . Since the 1-skeleton is homotopic to a bouquet of circles, there exists at least one such function  $f$  (indeed, we can choose a maximal tree  $T$  containing  $*$  in the 1-skeleton, put  $f(\sigma^1) = id_X$  for every edge  $\sigma^1$  of  $T$ , and for each edge  $\sigma^1$  which is not in  $T$ ,  $\sigma^1$  (equipped with its orientation) defines, jointly with  $T$ , an element  $\gamma \in \pi_1 \Sigma$ , and we put  $f(\sigma^1) = \rho(\gamma)$ ). Now, the boundary of a 2-cell  $\sigma_\alpha^2$  is a loop  $\gamma(\sigma_\alpha^2)$  (not based at  $*$ ) in the 1-skeleton, homotopically trivial in  $\Sigma$ . We can again write it as a word  $(\sigma_1^1)^{s_1} \cdots (\sigma_k^1)^{s_k}$ ,  $s_j = \pm 1$ , well defined up to cyclic permutation, and, by proposition 2.29 and by the fact that  $h$  is central in  $\widetilde{Ord(X)}$ , there exists  $n(\sigma_\alpha^2) \in \mathbb{Z}$  such that  $f(\gamma(\sigma_\alpha^2)) = h^{n(\sigma_\alpha^2)}$ . Finally, if  $c = \Sigma \lambda_i \sigma_i^2$  is a 2-cycle, we put  $e(\rho) \cdot c = \Sigma \lambda_i n(\sigma_i^2)$ .

**Theorem 2.32** *For every 2-cycle  $c$ , the integer  $e(\rho) \cdot c$  depends only on  $(X, o)$  and on  $\rho$ ; this defines an element  $e(\rho) \in H^2(\Sigma, \mathbb{Z})$ , called the Euler class of the representation  $\rho$ .*

If the surface  $\Sigma$  is oriented, then the evaluation of  $e(\rho)$  on the fundamental class is an integer, which we still call (abusively) the Euler class of the representation,  $e(\rho) \in \mathbb{Z}$ .

**Proof**

Step 1: we first prove that given base points  $* \in \Sigma$ ,  $x_0 \in X$  and a cellulation,  $e(\rho)$  does not depend on the choice of the function  $f$ . Let  $f_1$  and  $f_2$  two such functions. Choose a maximal tree  $T$  in the 1-skeleton, containing  $*$ . For every cell  $\sigma^1 \in T$ , put  $f'_1(\sigma^1) = f'_2(\sigma^1) = 1$ . Every cell  $\sigma^1$  which is not in  $T$  determines, jointly with  $T$ , a loop  $\gamma(\sigma^1)$  (since  $\sigma^1$  is equipped with an orientation) in the 1-skeleton, based at  $*$ . For  $i = 1, 2$  we put  $f'_i(\sigma^1) = f_i(\gamma(\sigma^1))$ . Since these elements  $\sigma^1 \notin T$  generate the fundamental group of the 1-skeleton, for every loop  $\gamma$  in the 1-skeleton we thus have  $f_i(\gamma) = f'_i(\gamma)$ , in particular  $f'_i$  projects on  $\rho$  and defines the same Euler class as  $f_i$ . Now, for all 1-cell  $\sigma^1$ , there exists an integer  $n(\sigma^1) \in \mathbb{Z}$  such that  $f'_1(\sigma^1) \circ f'_2(\sigma^1)^{-1} = h^{n(\sigma^1)}$ : this is 0 if  $\sigma^1 \in T$ , and otherwise this follows from proposition 2.29. Then for all 2-cell  $\sigma_\alpha^2$ , we have  $f_1(\gamma(\sigma_\alpha^2)) - f_2(\gamma(\sigma_\alpha^2)) = \sum_{\sigma^1 \in \partial \sigma^2} n(\sigma^1)$  (indeed, if  $\gamma(\sigma_\alpha^2)$

is given by the cyclic word  $(\sigma_1^1)^{s_1} \dots (\sigma_k^1)^{s_k}$  then

$$\begin{aligned} f_1(\gamma(\sigma_\alpha^2)) &= f_1(\sigma_1^1)^{s_1} \circ \dots \circ f_1(\sigma_k^1)^{s_k} = \left( f_2(\sigma_1^1)^{s_1} \circ h^{s_1 n(\sigma_1^1)} \right) \dots \left( f_2(\sigma_k^1)^{s_k} \circ h^{s_k n(\sigma_k^1)} \right) \\ &= f_2(\gamma(\sigma_\alpha^2)) \circ h^{\sum_{\sigma^1 \in \partial \sigma^2} n(\sigma^1)}, \end{aligned}$$

since  $h$  is central). Therefore, for every 2-cycle  $c$ , we have

$$e(\rho)_1 \cdot c - e(\rho)_2 \cdot c = \sum_{\sigma^2 \in c} \sum_{\sigma^1 \in \partial \sigma^2} n(\sigma^1),$$

where  $e(\rho)_i$  is the Euler class defined by the function  $f_i$ . This sum is zero, since  $c$  has no boundary.

Step 2: we prove that  $e(\rho)$  depends neither on the point  $* \in \Sigma$ , nor on the cellulation, nor on the base point  $x_0 \in X$ . Independence with respect to  $*$  follows from the fact that  $h$  is central in  $\text{Ord}(X)$ , so that no global conjugation of a representation can change its Euler class. Moreover,  $e(\rho)$  is defined in terms of the cellulation of  $\Sigma$ , and hence it is invariant under any isotopy of the cellulation of the surface. Now, let  $C = C_0 \cup C_1 \cup C_2$  be a cellulation of  $\Sigma$ . Let  $\sigma^1$  be a 1-cell, and  $x$  a point of  $\sigma^1$  which is not in  $C_0$ : we define a new cellulation  $C' = C'_0 \cup C'_1 \cup C'_2$ , with  $C'_0 = C_0 \cup \{x\}$ ,  $C'_1 = (C_1 \setminus \{\sigma^1\}) \cup \{\sigma_1^1, \sigma_2^1\}$  and  $C'_2 = C_2$  cutting the 1-cell  $\sigma^1$  into two cells along the vertex  $x$ ; the new edges  $\sigma_1^1$  and  $\sigma_2^1$  inherit of the orientation of  $\sigma^1$ . Then the Euler class defined by  $C'$  is the same as the one defined by  $C$ . Indeed, if  $f : C_1 \rightarrow \widetilde{\text{Ord}(X)}$  is a function which projects on  $\rho$ , define simply  $f' : C'_1 \rightarrow \widetilde{\text{Ord}(X)}$  by  $f'(\sigma_1^1) = 1$ ,  $f'(\sigma_2^1) = f(\sigma^1)$ , and  $f'(\sigma_\alpha^1) = f(\sigma_\alpha^1)$  for all the other 1-cells  $\sigma_\alpha^1$ . Similarly, if  $\sigma^2$  is a 2-cell, we define a new cellulation  $C'' = C''_0 \cup C''_1 \cup C''_2$  with  $C''_0 = C_0$ ,  $C''_1 = C_1 \cup \{\sigma_n^1\}$  and  $C''_2 = (C_2 \setminus \{\sigma^2\}) \cup \{\sigma_1^2, \sigma_2^2\}$  obtained by cutting the 2-cell  $\sigma^2$  into two, along the new edge  $\sigma_n^1$ , whose ends are vertices in  $\partial C_2$ . Let then  $f : C_1 \rightarrow \widetilde{\text{Ord}(X)}$  be

a function which projects on  $\rho$ . The loop defined by  $\partial\sigma_1^2$  (maybe with the opposite orientation) can be written as a word  $\sigma_n^1\gamma$  in the elements of  $C_1''$  (where  $\gamma$  stands for a word in the elements of  $\partial\sigma^2$ ). We thus put  $f'(\sigma_n^1) = f(\gamma)$ , and  $f'(\sigma_\alpha^1) = f(\sigma_\alpha^1)$  for all the other 1-cells  $\sigma_\alpha^1$ . Then by construction,  $f'$  projects on  $\rho$ . And  $f(\gamma(\sigma^2)) = f'(\gamma(\sigma^2)) = f'(\gamma(\sigma_1^2)) \circ f'(\gamma(\sigma_2^2))$ , hence  $f'$  defines the same Euler class as  $f$ . In particular, the Euler class defined by  $C''$  is still the same as the one defined by  $C$ . Now, we use the classical fact that by using these two operations (consisting in refining  $C$ ), as well as taking homotopies, from any pair of cellulations  $C^1$  and  $C^2$  of a same surface  $\Sigma$ , we can find a common refinement  $C^3$  of  $C^1$  and  $C^2$ . Finally, let  $x_0, x'_0$  be two points in  $X$ . Take a cellulation  $C$  of  $\Sigma$  and a function  $f : C_1 \rightarrow \widetilde{Ord(X)}_{x_0}$  which projects on  $\rho$ . Define then  $f' : C_1 \rightarrow \widetilde{Ord(X)}_{x'_0}$  by  $f' = F_{x_0x'_0} \circ f \circ (F_{x_0x'_0})^{-1}$ . Clearly,  $f'$  projects on  $\rho$  and  $f'$  defines the same Euler class as  $f$ , since  $h$  commutes with  $F_{x_0x'_0}$ .  $\square$

**Remark 2.33** *Let  $\Sigma$  be an oriented surface,  $\rho$  a representation and  $h : \Sigma' \rightarrow \Sigma$  a covering of degree  $d$ . Then  $e(\rho \circ h) = d \cdot e(\rho)$ . Indeed, it suffices to take a cellulation  $C$  of  $\Sigma$  which lifts to a cellulation of  $\Sigma'$ , to take a function  $f : C_1 \rightarrow Ord(X)$  which projects on  $\rho$ , and to take  $f' = f \circ h$  in order to define the Euler classes of  $\rho$  and  $\rho \circ h$  in  $\Sigma$  and  $\Sigma'$ .*

### 2.3.4 Milnor's algorithm

Consider a closed, oriented surface of genus  $g$ ,  $\Sigma_g$ , and equip it with a “standard” cellulation, featuring 1 single 2-cell, a single vertex and  $4g$  edges labelled by  $a_i, b_i, a_i^{-1}$  and  $b_i^{-1}$ , for  $1 \leq i \leq g$ . It yields a standard presentation  $\pi_1\Sigma_g = \langle a_1, \dots, b_g | \prod_i [a_i, b_i] = 1 \rangle$ . Then a representant of the fundamental class  $c \in H_2(\pi_1\Sigma_g, \mathbb{Z})$  is the 2-cycle consisting of the unique 2-cell, equipped with its orientation.

Given a representation  $\rho \in Hom(\pi_1\Sigma_g, Ord(X, o))$ , take some  $x_0 \in X$  and choose an arbitrary lift  $\widetilde{\rho(x)}$  for all  $x \in \{a_1, b_1, \dots, a_g, b_g\}$ . As we said before, we still denote by  $e(\rho) \in \mathbb{Z}$  the evaluation of  $e(\rho) \in H^2(\Sigma_g, \mathbb{Z})$  on the fundamental class  $c$ . Thus, by construction of the Euler class we have

$$[\widetilde{\rho(a_1)}, \widetilde{\rho(b_1)}] \cdots [\widetilde{\rho(a_g)}, \widetilde{\rho(b_g)}] = h^{e(\rho)}. \quad (1)$$

Since  $h$  is central in  $Ord(X, o, x_0)$  and commutes with  $F_{x_0x_1}$  for all  $x_1 \in X \setminus \{x_0\}$ , the result of this product of commutators does not depend on  $x_0$  neither on the choices of the lifts  $\widetilde{\rho(a_i)}, \widetilde{\rho(b_i)}$ .

### 2.3.5 Representations in $PSL(2, \mathbb{R})$

The Lie group  $PSL(2, \mathbb{R})$  is homeomorphic to a solid torus, and  $\pi_1(PSL(2, \mathbb{R})) \cong \mathbb{Z}$ . For all  $A \in PSL(2, \mathbb{R})$ , taking a lift  $\tilde{A} \in \widetilde{PSL(2, \mathbb{R})}$  amounts to lifting the homeomorphism  $A \in \text{Homeo}^+(\mathbb{S}^1)$  to a homeomorphism of the uni-

versal cover of the circle,  $\widetilde{\mathbb{S}^1} = \mathbb{R}$  (here we are using the natural injection  $PSL(2, \mathbb{R}) \hookrightarrow \text{Homeo}^+(\mathbb{S}^1)$ ). In other words, we have the following short exact sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{Z} & \rightarrow & \widetilde{PSL(2, \mathbb{R})} & \rightarrow & PSL(2, \mathbb{R}) \rightarrow 1. \\ & & & & \cap & & \cap \\ & & & & \text{Homeo}^+(\mathbb{R}) & & \text{Homeo}^+(\mathbb{S}^1) \end{array}$$

In this diagram, the sign of the generator of  $\mathbb{Z}$  is determined by the choice of the orientation of  $\mathbb{S}^1$ . As in [Gol88], we are going to denote by  $z \in \widetilde{PSL(2, \mathbb{R})}$  the image of this generator  $1 \in \mathbb{Z}$ .

Of course, all the construction of the Euler class applies here and if  $\widetilde{\rho(a_i)}$ ,  $\widetilde{\rho(b_i)}$  are arbitrary lifts of  $\rho(a_i)$ ,  $\rho(b_i)$  to  $\widetilde{PSL(2, \mathbb{R})}$ , then  $e(\rho) \in \mathbb{Z}$  is given by the following formula:

$$[\widetilde{\rho(a_1)}, \widetilde{\rho(b_1)}] \cdots [\widetilde{\rho(a_g)}, \widetilde{\rho(b_g)}] = z^{e(\rho)}.$$

In particular, notice that in the intermediate cover  $SL(2, \mathbb{R})$  of  $PSL(2, \mathbb{R})$ , the image of  $z$  is  $-\text{Id}$ . Hence, a representation  $\rho \in \text{Hom}(\pi_1 \Sigma_g, PSL(2, \mathbb{R}))$  lifts to  $SL(2, \mathbb{R})$  if and only if  $e(\rho)$  is even.

### 2.3.6 Another definition

The construction on the Euler class that we have just given is oriented towards the Milnor algorithm, which is essential for the technical propositions proved in the next section, which will be useful in the sequel. This algorithm also gives much of the intuition of how the Euler class behaves and how it can be computed. But the “true” definition of the Euler class is in terms of the (bounded) group cohomology of  $\Gamma$ , and we give it here. We refer to [ML95, Ghy87a, Thu88] for more details, and to [Cal04] for a detailed overview.

Let  $\Gamma$  be a group. One forms the *homogeneous complex* of  $\Gamma$  as follows. We denote by  $C_n(\Gamma)$  the free abelian group generated by the equivalence classes of the  $(n+1)$ -tuples of  $\Gamma$ , under the diagonal action of  $\Gamma$ : for all  $\gamma \in \Gamma$ ,  $(\gamma_0, \dots, \gamma_n) \sim (\gamma\gamma_0, \dots, \gamma\gamma_n)$ . The boundary operator is defined as

$$\partial(\gamma_0, \dots, \gamma_n) = \sum_{i=0}^n (-1)^i (\gamma_0, \dots, \hat{\gamma}_i, \dots, \gamma_n),$$

where  $(\gamma_0, \dots, \hat{\gamma}_i, \dots, \gamma_n)$  is the  $n$ -tuple formed by forgetting the  $(i+1)$ -th term  $\gamma_i$ . If  $A$  is a ring, the homology of the dual complex  $\text{Hom}(C_*(\Gamma), A)$  is denoted  $H^*(\gamma; A)$ ; it is the *cohomology of the group*  $\Gamma$ . This cohomology of  $\Gamma$  identifies to that of  $K(\Gamma, 1)$ .

Now let  $(X, o)$  be a cyclically ordered set, and let  $\rho: \Gamma \rightarrow \text{Ord}(X, o)$  be a representation. We then define a 2-cocycle  $c$  by letting  $c(\gamma_1, \gamma_2, \gamma_3) = o(\rho(\gamma_1), \rho(\gamma_2), \rho(\gamma_3))$ .

If  $\Gamma = \pi_1 \Sigma_g$  is the fundamental group of the (compact, closed, orientable) surface of genus  $g$ , then we have  $[c] = 2e(\rho)$ . The following remark follows:

**Remark 2.34** Let  $\Gamma$  be a group such that  $H^2(\Gamma, \mathbb{Z}) = 0$ , and let  $\rho_1: \pi_1 \Sigma_g \rightarrow \Gamma$ ,  $\rho_2: \Gamma \rightarrow \text{Ord}(X, o)$ . Then  $e(\rho_1 \circ \rho_2) = 0$ .

### 2.3.7 Finite sets suffice

Denote by  $\mathbb{F}_{2g}$  the free group on the set  $\{a_1, b_1, \dots, a_g, b_g\}$ , and  $w = [a_1, b_1] \cdots [a_g, b_g]$ . The images under the canonical surjection  $\mathbb{F}_{2g} \rightarrow \pi_1 \Sigma_g$  of the subwords of  $w$  form a set  $P$ , and in all the sequel of this text we denote by  $P_{ref}$  the set  $P \cup P^{-1}$ . A major interest of Milnor's algorithm is that we need only finitely many informations, concerning action of the finite set  $P_{ref}$  on the ordered set  $X$ , in order to be able to compute the Euler class of a representation.

This is the key idea which will prove, in chapter 3, that the Euler class extends *continuously* to the boundary of  $m_g^o$ .

More precisely, the idea is the following:

**Proposition 2.35** Let  $(X, o)$  and  $(X', o')$  be two cyclically ordered sets, equipped with "base points"  $x_0, x'_0$ . Let  $\rho: \pi_1 \Sigma_g \rightarrow \text{Ord}(X, o)$  and  $\rho': \pi_1 \Sigma_g \rightarrow \text{Ord}(X', o')$  be two representations. Suppose that for all  $g_1, g_2, g_3 \in P_{ref}$ ,

$$o(g_1 x_0, g_2 x_0, g_3 x_0) = o'(g_1 x'_0, g_2 x'_0, g_3 x'_0).$$

Then  $e(\rho_1) = e(\rho_2)$ .

In fact we shall need a slightly more subtle statement, since we want the Euler class to be stable under small degenerations. We shall leave the proof of proposition 2.35 as an (easy) exercise, and instead we will prove the following:

**Proposition 2.36** Let  $(X, o)$  and  $(X', o')$  be two cyclically ordered sets, equipped with "base points"  $x_0, x'_0$ . Let  $\rho: \pi_1 \Sigma_g \rightarrow \text{Ord}(X, o)$  and  $\rho': \pi_1 \Sigma_g \rightarrow \text{Ord}(X', o')$  be two representations. Let  $y_0 \in X$  and  $y'_0 \in X'$ . Suppose also that  $x_0 \notin P_{ref} \cdot y_0$ , that  $\text{card}(P_{ref} \cdot y_0) \geq 2$ , and that for all  $g_1, g_2, g_3 \in P_{ref}$ ,

$$o(g_1 x_0, g_2 x_0, g_3 y_0) = 1 \Rightarrow o'(g_1 x'_0, g_2 x'_0, g_3 y'_0) = 1$$

and

$$o(g_1 x_0, g_2 y_0, g_3 y_0) = 1 \Rightarrow o'(g_1 x'_0, g_2 y'_0, g_3 y'_0) = 1.$$

Then  $e(\rho_1) = e(\rho_2)$ .

Everything relies on the two following elementary lemmas:

**Lemma 2.37** Let  $(X, o)$  be a cyclically ordered set, and  $f \in \text{Ord}(X)$ . Suppose we have a base point  $x_0 \in X$  and an element  $y \in X \setminus \{x_0\}$  such that  $f(y) \neq x_0$ . Denote by  $\tilde{f}$  the lift of  $f$  to  $\mathbb{Z} \times X$  verifying  $\tilde{f}(0, x_0) = (0, f(x_0))$ , and denote by  $n$  the integer such that  $\tilde{f}(0, y) = (n, f(y))$ . Then  $n$  depends only on  $o(x_0, f(x_0), f(y))$ . More precisely,  $n = \max(0, -o(x_0, f(x_0), f(y)))$ .



**Lemma 2.38** *Let  $(X, o)$  be a cyclically ordered set,  $f \in \text{Ord}(X)$  and  $x_0 \in X$ . Let  $x_1, x_2 \in X$  be such that  $o(x_0, x_1, x_2) = o(f(x_0), x_1, x_2) = 1$ . Then there exists a lift  $\tilde{f}$  of  $f$  to  $\mathbb{Z} \times X$  such that  $(-1, x_2) <_{x_0} \tilde{f}(0, x_0) <_{x_0} (0, x_1)$ . Moreover, if  $y \in X$  is such that  $o(x_1, x_2, y) \leq 0$  and  $o(x_1, x_2, f(y)) \leq 0$ , then this lift  $\tilde{f}$  verifies  $\tilde{f}(0, y) = (0, f(y))$ .*

**Proof** of lemma 2.37.

We have  $f(y) \neq f(x_0)$  hence  $y \neq x_0$  and hence  $(0, x_0) <_{x_0} (0, y) <_{x_0} (1, x_0)$ . The function  $\tilde{f}$  is increasing, so that  $(0, f(x_0)) <_{x_0} (n, f(y)) <_{x_0} (1, f(x_0))$ .

If  $o(x_0, f(x_0), f(y)) = 0$  then  $x_0 = f(x_0)$  and hence  $n = 0$ .

If  $o(x_0, f(x_0), f(y)) = 1$  then  $(0, f(x_0)) <_{x_0} (0, f(y)) <_{x_0} (1, f(x_0))$  and then  $n = 0$ .

If  $o(x_0, f(x_0), f(y)) = -1$  then  $(-1, f(x_0)) <_{x_0} (0, f(y)) <_{x_0} (0, f(x_0))$  and in that case  $n = 1$ .  $\square$

**Proof** of lemma 2.38.

There are three cases to consider here.

- If  $f(x_0) = x_0$ , then  $(-1, x_2) <_{x_0} (0, f(x_0)) <_{x_0} (0, x_1)$ , so we take  $\tilde{f}$  such that  $\tilde{f}(0, x_0) = (0, f(x_0))$ . We then have  $(0, x_0) <_{x_0} (0, y) <_{x_0} (1, x_0)$ , hence, by applying  $\tilde{f}$  (which is strictly increasing):  $(0, x_0) <_{x_0} \tilde{f}(0, y) <_{x_0} (1, x_0)$ , so that  $\tilde{f}(0, y) = (0, f(y))$ .
- If  $o(x_0, x_1, f(x_0)) = -1$  then  $(-1, x_2) <_{x_0} (0, x_0) <_{x_0} (0, f(x_0)) <_{x_0} (0, x_1)$  so we take again  $\tilde{f}$  such that  $\tilde{f}(0, x_0) = (0, f(x_0))$ . By lemma 2.37, it suffices to prove that  $o(x_0, f(x_0), f(y)) \geq 0$  in order to have  $n = 0$  and thus  $\tilde{f}(0, y) = (0, f(y))$ . But if  $o(x_0, f(y), f(x_0)) = 1$ , since  $o(x_0, f(x_0), x_1) = 1$  we get  $o(x_0, f(y), x_1) = 1$  so  $o(x_1, x_0, f(y)) = 1$  which, together with  $o(x_1, x_2, x_0) = 1$ , gives  $o(x_1, x_2, f(y)) = 1$ , a contradiction.
- If  $o(x_0, x_1, f(x_0)) = 1$  then we have  $o(x_0, x_2, f(x_0)) = 1$  (indeed, this follows from the equalities  $o(f(x_0), x_0, x_1) = o(f(x_0), x_1, x_2) = 1$ ), hence

$$(-1, x_2) <_{x_0} (-1, f(x_0)) <_{x_0} (0, x_0) <_{x_0} (0, x_1)$$

and we take  $\tilde{f}$  such that  $\tilde{f}(0, x_0) = (-1, f(x_0))$ . In particular,  $\tilde{f} = \tilde{f}' \circ h^{-1}$ , where  $\tilde{f}'(0, x_0) = (0, f(x_0))$ . And we have  $o(x_0, f(x_0), f(y)) = -1$  (indeed, if  $f(y) = x_1$  or  $x_2$  we already have this equality, and otherwise  $o(x_2, x_1, f(y)) = 1$ , which, together with the equality  $o(x_2, x_0, x_1) = 1$ , gives  $o(x_2, x_0, f(y)) = 1$ , i.e.  $o(x_0, f(y), x_2) = 1$ , which, together with  $o(x_0, x_2, f(x_0)) = 1$ , yields  $o(x_0, f(y), f(x_0)) = 1$ ) therefore lemma 2.37 applied to  $\tilde{f}'$  implies that  $\tilde{f}'(0, y) = (1, f(y))$ , whence  $\tilde{f}(0, y) = (0, f(y))$  once again.  $\square$

We shall now prove proposition 2.36. We stated proposition 2.35 only in order to set up the idea of proposition 2.36, and we shall not use it. The proof is similar, though a little easier, than the proof of proposition 2.36.

**Proof** of proposition 2.36.

Denote  $y_i = [\rho(a_{i+1}), \rho(b_{i+1})] \cdots [\rho(a_g), \rho(b_g)] \cdot y_0$ . Then in particular  $y_g = y_0$ . The integers  $m_i$  such that  $[\rho(a_i), \rho(b_i)](0, y_i) = (m_i, y_{i-1})$  do not depend on the choices of the lifts  $\widetilde{\rho(a_i)}$ ,  $\widetilde{\rho(b_i)}$ , and  $e(\rho) = \sum_{i=1}^g m_i$ , by Milnor's algorithm.

We also use the same notations in  $X'$ .

The finite set  $P_{ref} \cdot y_0 \subset X \setminus \{x_0\}$ , equipped with the order  $<_{x_0}$ , contains a smallest element  $x_1$  and a biggest element  $x_2$ . Similarly we define  $x'_1$  and  $x'_2$  in  $X'$ . Let then  $\gamma_1$  be an element of  $P_{ref}$  such that  $\gamma_1 y'_0 = x'_1$ . Then for all  $\gamma \in P_{ref}$ ,  $o'(x'_0, \gamma_1 y'_0, \gamma y'_0) \leq 0$ , hence  $o(x_0, \gamma_1 y_0, \gamma y_0) \leq 0$ , so that  $\gamma_1 y_0$  is minimal among  $P_{ref} \cdot y_0$  in  $X \setminus \{x_0\}$  for the order  $<_{x_0}$ , that is,  $\gamma_1 y_0 = x_1$ . Similarly,  $x_2$  and  $x'_2$  correspond to (at least) one same element  $\gamma_2 \in P_{ref}$ . Moreover, since  $\text{Card}(P_{ref} \cdot y_0) \geq 2$ , we have  $x_1 <_{x_0} x_2$ , hence  $o(x_0, x_1, x_2) = 1$ .

For every element  $\gamma \in \{a_1, b_1, \dots, a_g, b_g\}$  we define  $\widetilde{\rho(\gamma)}$  and  $\widetilde{\rho'(\gamma)}$  as follows. If  $\rho(\gamma) \cdot x_0 \neq x_0$ , we choose  $\widetilde{\rho(\gamma)}$  such that  $\widetilde{\rho(\gamma)}(0, x_0) = (0, \rho(\gamma) \cdot x_0)$ ; and we choose  $\widetilde{\rho'(\gamma)}$  such that  $\widetilde{\rho'(\gamma)}(0, x'_0) = (0, \rho'(\gamma) \cdot x'_0)$ . Otherwise, if  $\rho(\gamma) \cdot x_0 = x_0$  then we have  $o(x_0, x_1, x_2) = o(\rho(\gamma) \cdot x_0, x_1, x_2) = 1$  hence, by lemma 2.38,  $\rho(\gamma)$  possesses a lift  $\widetilde{\rho(\gamma)}$  such that  $(-1, x_2) <_{x_0} \widetilde{\rho(\gamma)}(0, x_0) <_{x_0} (0, x_1)$ , and in that case again we have  $o'(x'_0, x'_1, x'_2) = o'(\rho'(\gamma) \cdot x'_0, x'_1, x'_2) = 1$  (indeed,  $o(x_0, x_1, x_2) = o(x_0, \gamma_1 y_0, \gamma_2 y_0) = 1$  so that  $o'(x'_0, x'_1, x'_2) = 1$  and, similarly,  $o(\gamma x_0, x_1, x_2) = 1$  hence  $o'(\rho'(\gamma) x'_0, x'_1, x'_2) = 1$ , since  $\gamma, \gamma_1, \gamma_2 \in P_{ref}$ ) hence, still by applying lemma 2.38, we can define  $\widetilde{\rho'(\gamma)}$  in such a way that  $(-1, x'_2) <_{x'_0} \widetilde{\rho'(\gamma)}(0, x'_0) <_{x'_0} (0, x'_1)$ .

Now denote by  $n_{i_1}, n_{i_2}, n_{i_3}, n_{i_4}$  the integers such that

$$\begin{aligned} \widetilde{\rho(b_i)}^{-1}(0, y_i) &= (n_{i_4}, \rho(b_i)^{-1} \cdot y_i), \\ \widetilde{\rho(a_i)}^{-1} \cdot (0, \rho(b_i)^{-1} \cdot y_i) &= (n_{i_3}, \rho(a_i)^{-1} \rho(b_i)^{-1} \cdot y_i), \\ &\dots, \end{aligned}$$

and similarly we define integers  $n'_{i_1}, \dots, n'_{i_4}$ .

Let us first check that  $n_{i_4} = n'_{i_4}$ . Denote  $\gamma = [a_{i+1}, b_{i+1}] \cdots [a_g, b_g]$  (that is the element of  $\pi_1 \Sigma_g$  defined by the subword following  $b_{i-1}$  in  $w$ ).

- If  $\rho(b_i) \cdot x_0 \neq x_0$ , then  $o(x_0, \rho(b_i)^{-1} \cdot x_0, \rho(b_i^{-1} \gamma) \cdot y_0) \neq 0$  (indeed, these three points are all distinct since  $\gamma$  and  $b_i^{-1} \gamma$  are in  $P_{ref}$ ), and hence

$$o'(x'_0, \rho'(b_i)^{-1} \cdot x'_0, \rho'(b_i^{-1} \gamma) \cdot y'_0) = o(x_0, \rho(b_i)^{-1} \cdot x_0, \rho(b_i^{-1} \gamma) \cdot y_0)$$

so by lemma 2.37 (applied to  $f = \rho(b_i^{-1})$ ,  $y = y_i$  and to  $f = \rho'(b_i^{-1})$  and  $y = y'_i$ ) we have  $n_{i_4} = n'_{i_4}$ .

- If  $\rho(b_i) \cdot x_0 = x_0$ , then by lemma 2.38, this time we have  $n_{i_4} = n'_{i_4} = 0$ .

Similarly we get  $n_{i_3} - n_{i_4} = n'_{i_3} - n'_{i_4}, \dots, n_{i_1} - n_{i_2} = n'_{i_1} - n'_{i_2}$ , so that  $m_i = m'_i$ , and hence  $e(\rho) = e(\rho')$ .  $\square$

## 2.4 Rigidity of non discrete representations

In all this section,  $o$  will denote the natural cyclic order on  $\partial\mathbb{H}^2 = \mathbb{S}^1$ . The object of this section is to prove the following statement:

**Theorem 2.39** *Let  $\rho_1, \rho_2 \in R_g$  be two non discrete, non elementary representations, and let  $a_1, a_2 \in \partial\mathbb{H}^2$ . Suppose that for all  $\gamma_1, \gamma_2, \gamma_3 \in \pi_1\Sigma_g$ ,*

$$o(\rho_1(\gamma_1) \cdot a_1, \rho_1(\gamma_2) \cdot a_1, \rho_1(\gamma_3) \cdot a_1) = o(\rho_2(\gamma_1) \cdot a_2, \rho_2(\gamma_2) \cdot a_2, \rho_2(\gamma_3) \cdot a_2).$$

*Then there exists  $g \in PSL(2, \mathbb{R})$  such that  $\rho_1 = g\rho_2g^{-1}$ .*

The proof of theorem 2.39 depends on the following lemma:

**Lemma 2.40** *For any  $A \in PSL(2, \mathbb{R})$  and any  $a \in \partial\mathbb{H}^2$ :*

- (1)  *$o(A^{n_1}a, A^{n_2}a, A^{n_3}a) = 0$  for all  $(n_1, n_2, n_3) \in \mathbb{Z}^3$  if and only if  $a$  is a fixed point of  $A^2$ ,*
- (2) *if  $o(A^{n_1}a, A^{n_2}a, A^{n_3}a) = 1$  for all  $(n_1, n_2, n_3) \in \mathbb{Z}^3$  such that  $n_1 < n_2 < n_3$  then  $A$  is parabolic or hyperbolic,*
- (3) *similarly, if  $o(A^{n_1}a, A^{n_2}a, A^{n_3}a) = 1$  for all  $(n_1, n_2, n_3) \in \mathbb{Z}^3$  such that  $n_1 < n_2 < n_3$  then  $A$  is parabolic or hyperbolic,*
- (4) *otherwise,  $A$  is a non trivial elliptic element of  $PSL(2, \mathbb{R})$ , and the data of  $o(A^{n_1}a, A^{n_2}a, A^{n_3}a)$  determines  $A$  up to conjugation.*

### Proof

If  $A$  is parabolic or hyperbolic, then one of the cases (1), (2) or (3) arises depending on whether  $a$  is a fixed point of  $A$ , and on the direction in which  $A$  moves the region of  $\mathbb{S}^1$  minus the fixed point(s) of  $A$ , containing  $a$ . Conversely, it is clear that the cases (2) and (3) can happen only if  $A$  is parabolic or hyperbolic.

Now suppose that  $A$  is elliptic. Then, up to conjugating  $A$ , we have  $A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ , for some  $\theta \in [0, \pi)$ . In particular, we note that in this case, the function  $(n_1, n_2, n_3) \mapsto o(A^{n_1}a, A^{n_2}a, A^{n_3}a)$  does not depend on  $a$  any more, since  $A$  acts as a rotation of angle  $2\theta$  on the circle  $\mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ . Now remark that  $o(a, Aa, A^2a)$  equals 1 if  $2\theta \in (0, \pi)$ , it equals  $-1$  if  $2\theta \in (\pi, 2\pi)$  and it equals 0 otherwise. Similarly, if  $\frac{\theta}{\pi} = \frac{\alpha_1}{2} + \frac{\alpha_2}{4} + \dots + \frac{\alpha_n}{2^n} + \dots$  is the dyadic expression of  $\frac{\theta}{\pi}$  (we do not consider expressions with  $\alpha_n = 1$  for all  $n$  large enough, hence this expression is unique), then:

- if  $o(a, A^{2^{n-1}}a, A^{2^n}a) = 1$  then  $\alpha_n = 1$  and the expression does not stop at  $\alpha_n$  (i.e., one of the  $\alpha_{n+1}, \dots$ , is not zero),

- if  $o(a, A^{2^{n-1}}a, A^{2^n}a) = -1$  then  $\alpha_n = 0$  and the expression continues after  $\alpha_n$ ,
- if  $o(a, A^{2^{n-1}}a, A^{2^n}a) = 0$  then the angle of the rotation  $A^{2^{n-1}}$  is 0 or  $\pi$ , i.e.  $\alpha_k = 0$  for all  $k \geq n + 1$ .

In particular, the data of  $o(A^{n_1}a, A^{n_2}a, A^{n_3}a)$  for all  $(n_1, n_2, n_3) \in \mathbb{Z}^3$  determines the angle  $2\theta \in [0, 2\pi)$ , unless if  $2\theta = 0$  or  $\pi$  (since in both cases, this number is always 0).  $\square$

**Remark 2.41** *As we said in section 2.3.6, the cocycle we are computing here is twice the Euler class. In particular, lemma 2.40 is related to the fact that  $H_b^2(\mathbb{Z}, \mathbb{Z}) = \mathbb{R}/\mathbb{Z}$  (see [Ghy87b], proposition 3-1). Since the element  $\frac{1}{2}$  has order 2 in  $\mathbb{R}/\mathbb{Z}$ , and since we are computing twice the Euler class, lemma 2.40 does not distinguish the rotation of half the circle from the identity.*

**Proof** of theorem 2.39.

It is well-known (see e.g. [Kat92], exercise 2.14 or [Wol07], proposition 1.1.14 for a proof) that every non discrete, non elementary subgroup of  $PSL(2, \mathbb{R})$  contains an elliptic element of infinite order. Let  $\gamma \in \pi_1 \Sigma_g$  be such that  $\rho_1(\gamma)$  is elliptic of infinite order. Conjugate  $\rho_1$  so that  $\rho_1(\gamma) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ , for some irrational  $\theta \in [0, \pi)$ . Now lemma 2.40 implies that  $\rho_2(\gamma)$  is conjugate to  $\rho_1(\gamma)$ . Now let  $\gamma' \in \pi_1 \Sigma_g$ . Write  $\rho_1(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Then

$$\text{Tr}(\rho_1(\gamma^n \gamma')) = |(a + d) \cos(n\theta) + (b - c) \sin(n\theta)| \quad (2)$$

is the absolute value of the scalar product of the Euclidean plane vectors  $\begin{pmatrix} a + d \\ b - c \end{pmatrix}$  and  $\begin{pmatrix} \cos(n\theta) \\ \sin(n\theta) \end{pmatrix}$ . The latter can be chosen to be arbitrarily close to any direction, since  $\theta$  is irrational, and the former has norm greater or equal to 4 (since  $ad - bc = 1$ ), thus  $\text{Tr}(\rho_1(\gamma^n \gamma')) \in [\frac{1}{2}, \frac{3}{2}]$  for infinitely many values of  $n$ ; hence  $\rho_1(\gamma^n \gamma')$  is elliptic, of order different from 2, for infinitely many values of  $n$ . For these values it follows from lemma 2.40 that  $\text{Tr}(\rho_1(\gamma^n \gamma')) = \text{Tr}(\rho_2(\gamma^n \gamma'))$ , so that, from the formula (2) we derive that  $\text{Tr}(\rho_1(\gamma')) = \text{Tr}(\rho_2(\gamma'))$ . It now follows from proposition 2.15 that  $\rho_1$  and  $\rho_2$  are conjugate by an element of  $Isom(\mathbb{H}^2)$ , but a conjugation reversing the orientation would change  $\rho_1(\gamma)$  into a rotation of angle  $-\theta$  instead of  $\theta$ , hence  $\rho_1$  and  $\rho_2$  are conjugate by an element of  $PSL(2, \mathbb{R})$ .  $\square$

## 2.5 Almost faithful morphisms

The connectedness of  $\overline{m}_g^u$  (theorem 3.36) strongly relies on a property of surface groups related to the fact that these groups are “limit groups” (see e.g. [Sel01, Gui04, CG05]).

Let us fix some notations first. Fix a standard presentation of the fundamental group

$$\pi_1 \Sigma_g = \langle a_1, \dots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

of the surface  $\Sigma_g$ . The set  $S = \{a_1, \dots, b_g\}$  generates  $\pi_1 \Sigma_g$ , and we denote by  $\mathbb{F}$  the free group on this generating set; and denote by  $\pi: \mathbb{F} \rightarrow \pi_1 \Sigma_g$  the canonical surjection. Denote by  $B_n \subset \pi_1 \Sigma_g$  the ball of centre 1 and radius  $n$  for the Cayley metric associated to the generating set  $S$ . Finally, we denote by  $\mathbb{F}_k$  a free group of rank  $k$ .

A group  $\Gamma$  is said to be *residually free* if for all  $\gamma \in \Gamma \setminus \{1\}$ , there exists a morphism  $\varphi: \Gamma \rightarrow \mathbb{F}_2$  such that  $\varphi(\gamma) \neq 1$ . We say that  $\Gamma$  is *fully residually free* if for every finite subset  $\{\gamma_1, \dots, \gamma_n\} \subset \Gamma \setminus \{1\}$ , there exists a morphism  $\varphi: \Gamma \rightarrow \mathbb{F}_2$  such that for all  $i \in \{1, \dots, n\}$ ,  $\varphi(\gamma_i) \neq 1$ . We will use the following celebrated result:

**Theorem 2.42 (G. Baumslag [Bau62])** *For all  $g \geq 2$ , the group  $\pi_1 \Sigma_g$  is fully residually free. In other words, for every  $n \geq 0$ , there exists a morphism  $\varphi_n: \pi_1 \Sigma_g \rightarrow \mathbb{F}_2$  such that  $\ker(\varphi_n) \cap B_n = \{1\}$ .*

Heuristically, the morphisms  $\varphi_n$  are “more and more injective”. In the language of [Sel01, CG05], the group  $\pi_1 \Sigma_g$  is a “limit group” of the group  $\mathbb{F}_2$ . In fact, we will need a statement a little more general: we will need to make two explicitly given morphisms “more and more injective”, by composing them with automorphisms of  $\pi_1 \Sigma_g$ . For all  $g \geq 3$ , we denote  $e_g: \pi_1 \Sigma_g \rightarrow \pi_1 \Sigma_{g-1}$  the morphism consisting of collapsing the last handle. More precisely, given the two standard presentations

$$\pi_1 \Sigma_g = \left\langle a_1, \dots, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \right\rangle$$

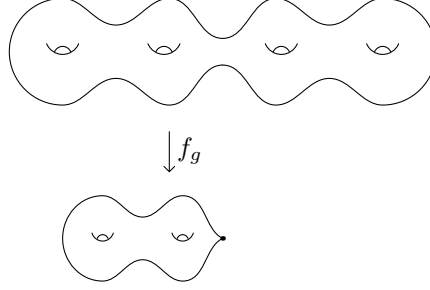
and

$$\pi_1 \Sigma_{g-1} = \left\langle a_1, \dots, b_{g-1} \mid \prod_{i=1}^{g-1} [a_i, b_i] = 1 \right\rangle,$$

the map  $e_g$  is defined by  $e_g(\gamma) = \gamma$  for  $\gamma = a_1, b_1, \dots, a_{g-1}, b_{g-1}$  and  $e_g(\gamma) = 1$  for  $\gamma = a_g, b_g$ . Now let  $g \geq 2$ ; we will consider a Fuchsian group  $G_g$  and a morphism  $p_g: \pi_1 \Sigma_g \rightarrow G_g$ , depending on the parity of  $g$ .

If  $g$  is even,  $g = 2g'$ , we fix a Fuchsian group  $G_g$  of signature  $(g'; 2)$ . It is the fundamental group of the hyperbolic orbifold of genus  $g'$  and with one conic singularity of angle  $\pi$  ( $= \frac{2\pi}{2}$ ). This orbifold possesses a covering  $f_g$  of

degree 2, branched over the conic singularity,



and we set  $p_g = f_{g*}: \pi_1 \Sigma_g \rightarrow G_g$ . Recall that the Fuchsian group  $G_g$ , of signature  $(g'; 2)$ , has the following presentation:

$$G_g = \langle \alpha_1, \dots, \beta_{g'} \mid ([\alpha_1, \beta_1] \cdots [\alpha_{g'}, \beta_{g'}])^2 = 1 \rangle.$$

Then we define the map  $p_g: \pi_1 \Sigma_g \rightarrow G_g$  by letting  $p_g(a_i) = p_g(a_{g'+i}) = \alpha_i$  and  $p_g(b_i) = p_g(b_{g'+i}) = \beta_i$  for all  $i$  between 1 and  $g'$ .

In the case when  $g$  is odd,  $g = 2g' + 1$ , we fix a Fuchsian group  $G_g$  of signature  $(g'; 2, 2, 2)$ . Recall that such a group has the following presentation:

$$G_g = \langle q_1, q_2, \alpha_1, \dots, \beta_{g'} \mid q_1^2 = q_2^2 = 1, (q_1 q_2 [\alpha_1, \beta_1] \cdots [\alpha_{g'}, \beta_{g'}])^2 = 1 \rangle.$$

We denote by  $p_g: \pi_1 \Sigma_g \rightarrow G_g$  the morphism defined by  $p_g(a_1) = q_1^{-1}$ ,  $p_g(b_1) = q_2^{-1}$ , and  $p_g(a_i) = (q_1 q_2)^{-1} \alpha_{i-1} (q_1 q_2)$ ,  $p_g(b_i) = (q_1 q_2)^{-1} \beta_{i-1} (q_1 q_2)$ , for all  $i$  between 2 and  $g' + 1$ , and  $p_g(a_i) = \alpha_{i-g'-1}$  and  $p_g(b_{i-g'-1}) = \beta_i$  when  $g' + 2 \leq i \leq g$ . It is a little more delicate, in the case when  $g$  is odd, to draw  $p_g$  as a covering.

In the two cases of parity of  $g$ , it has been proved (see [FW07], proposition 4.5) that the discrete representation  $p_g$  (as an element of  $R_g$ ) has Euler class  $2g - 3$ ; this is why we consider it.

**Lemma 2.43** *Let  $g \geq 4$ . Then for all  $n \geq 0$ , there exists an element  $\gamma_n \in \text{Aut}(\pi_1 \Sigma_g)$  such that the kernel of the morphism  $p_g \circ \gamma_n: \pi_1 \Sigma_g \rightarrow G_g$  does not contain any non-trivial element of length less than  $n$ .*

In other words, we can “mix” the curves representing  $a_i$  and  $b_i$  by Dehn twists, so that the map  $p_g$  does not kill any word of given length, that is, we can change this map into maps which are “arbitrarily injective”.

Similarly:

**Lemma 2.44** *Let  $g \geq 3$ . Then for all  $n \geq 0$ , there exists an element  $\gamma_n \in \text{Aut}(\pi_1 \Sigma_g)$  such that the kernel of the map  $e_g \circ \gamma_n: \pi_1 \Sigma_g \rightarrow G_g$  does not contain any non-trivial elements of length less than  $n$ .*

V. Guirardel pointed out to me the following proof, which is due to Z. Sela.

**Proposition 2.45** *Let  $\Sigma$  be a surface, possibly with boundary, of Euler characteristic less or equal to  $-1$ . Let  $\varphi: \pi_1 \Sigma \rightarrow \mathbb{F}$  be a morphism of non abelian image in a free group  $\mathbb{F}$ , whose restrictions to the fundamental groups of the boundary components are injective. Then for all finite subset  $P \subset \pi_1 \Sigma \setminus \{1\}$ , there exists a diffeomorphism  $\gamma_P$  of  $\Sigma$ , preserving pointwise the boundary components, and such that  $\text{Ker}(\varphi \circ \gamma_{P*}) \cap P = \emptyset$ .*

**Corollary 2.46** *Let  $g \geq 2$ , and let  $\varphi: \pi_1 \Sigma_g \rightarrow \mathbb{F}$  be a morphism of non abelian image. Then for all  $n$ , there exists  $\gamma_n \in \text{Aut}(\pi_1 \Sigma_g)$  such that  $\text{Ker}(\varphi \circ \gamma_n) \cap B_n = \{1\}$ .*

The lemmas 2.43 and 2.44 follow:

**Proof** of lemma 2.43.

The map  $\varphi_g: G_g \rightarrow \mathbb{F}_2$  defined by  $\varphi_g(a_1) = x$ ,  $\varphi_g(a_2) = y$  and  $\varphi_g(u) = 1$  for all the other generators  $u$  of  $G_g$ , with  $\mathbb{F}_2 = \langle x, y \rangle$ , is a morphism of non abelian image, and  $\varphi_g \circ p_g: \pi_1 \Sigma_g \rightarrow \mathbb{F}_2$  is therefore a morphism satisfying the hypotheses of corollary 2.46. Thus, we can conjugate it by automorphisms of  $\pi_1 \Sigma_g$  in order to make it “arbitrarily injective”.  $\square$

**Proof** of lemma 2.44.

If  $g \geq 3$ , the map  $\varphi_g: \pi_1 \Sigma_{g-1} \rightarrow \mathbb{F}_2$  defined by  $\varphi_g(a_1) = x$ ,  $\varphi_g(a_2) = y$  and  $\varphi_g(u) = 1$  for all the other generators  $\pi_1 \Sigma_{g-1}$  is still a morphism of non abelian image.  $\square$

The proof of the proposition relies on the following two lemmas.

**Lemma 2.47 (Z. Sela [Sel01], lemma 5.13)** *Let  $\Sigma$  be a surface, possibly with boundary, of Euler characteristic less or equal to  $-1$ . Let  $\varphi: \pi_1 \Sigma \rightarrow \mathbb{F}$  be a morphism of non abelian image in a free group  $\mathbb{F}$ , whose restrictions to the fundamental groups of the boundary components are injective. Then there exists a family of disjoint closed simple curves  $c_1, \dots, c_p$  in  $\Sigma$ , which cut  $\Sigma$  into pairs of pants, and such that the restriction of  $\varphi$  to the fundamental group of each pair of pant is injective.*

**Lemma 2.48 (G. Baumslag [Bau62], proposition 1)** *Let  $\mathbb{F}$  be a free group and let  $a_1, \dots, a_n, c \in \mathbb{F}$  be such that  $c$  does not commute with any of the  $a_i$ ’s. Then for all  $k_0, \dots, k_n$  large enough, the element  $c^{k_0} a_1 c^{k_1} a_2 \dots c^{k_{n-1}} a_n c^{k_n}$  is non trivial in  $\mathbb{F}$ .*

**Proof** of proposition 2.45.

Denote by  $\chi(\Sigma)$  and  $g(\Sigma)$  the Euler characteristic and the genus of  $\Sigma$ . We shall work by induction on  $(-\chi(\Sigma), g(\Sigma))$ , following the lexicographic order. The number  $g(\Sigma)$  being between 0 and  $1 - \frac{\chi(\Sigma)}{2}$ , the element  $(-\chi(\Sigma), g(\Sigma))$  indeed describes a set which is in bijection preserving the order with  $\mathbb{N}$ .

If  $\chi(\Sigma) = -1$ , then  $\varphi$  is injective (see e.g. [CG05], proposition 3.1). Hence, suppose that the proposition is true for all  $\Sigma'$  such that  $(-\chi(\Sigma'), g(\Sigma')) <$

$(-\chi(\Sigma), g(\Sigma))$  (for the lexicographic order) and consider curves  $c_1, \dots, c_p$  as in Z. Sela's lemma.

Suppose first that  $c_1$  is a separating curve: denote  $\Sigma = \Sigma_1 \cup_{c_1} \Sigma_2$ . Put the base point near  $c_1$ , on the side of  $\Sigma_1$ . We have  $\pi_1 \Sigma = \pi_1 \Sigma_1 *_{\alpha} \pi_1 \Sigma_2$ , where  $\alpha$  is represented by the curve  $c_1$ , deformed so that it passes through the base point. Fix a finite subset  $P \subset \pi_1 \Sigma$ . For every  $m \in P$ , choose a writing  $m = a_1 \alpha^{k_1} b_1 \alpha^{l_1} \dots a_n \alpha^{k_n} b_n \alpha^{l_n}$ , with  $a_i \in \pi_1 \Sigma_1$  and  $b_i \in \pi_1 \Sigma_2$ , and such that  $a_i, b_i$  do not commute with  $\alpha$  (except maybe  $a_1$  or  $b_n$ , in which case we do not write them in  $m$ ). Denote by  $P_1$  the subset of  $\pi_1 \Sigma_1$  defined by the elements  $\alpha a_i \alpha^{-1} a_i^{-1}$  and denote by  $P_2$  the subset of  $\pi_1 \Sigma_2$  defined by the elements  $\alpha b_i \alpha^{-1} b_i^{-1}$ . By induction hypothesis, there exists a diffeomorphism  $\gamma_1$  of  $\Sigma_1$  fixing the boundary of  $\Sigma_1$  (as well as the boundary of the curve  $c_1$ ), and a diffeomorphism  $\gamma_2$  of  $\Sigma_2$  fixing the boundary of  $\Sigma_2$  such that for all  $u \in P_1$  we have  $\varphi \circ \gamma_{1*}(u) \neq 1$  and such that for all  $u \in P_2$  we have  $\varphi \circ \gamma_{2*}(u) \neq 1$ . Consider then a diffeomorphism  $\gamma_k: \Sigma \rightarrow \Sigma$  defined by  $\gamma_1$  on  $\Sigma_1$ ,  $\gamma_2$  on  $\Sigma_2$  and by  $k$  Dehn twists along  $c_1$ . Then  $\gamma_{k*}: \pi_1 \Sigma \rightarrow \pi_1 \Sigma$  is defined as follows: if  $a_1, \dots, a_n \in \pi_1 \Sigma_1$  and  $b_1, \dots, b_n \in \pi_1 \Sigma_2$ , we have

$$\gamma_{k*} \left( a_1 \alpha^{k_1} b_1 \alpha^{l_1} \dots a_n \alpha^{k_n} b_n \alpha^{l_n} \right) = \gamma_{1*}(a_1) \alpha^{k_1+k} \gamma_{2*}(b_1) \alpha^{l_1-k} \dots \gamma_{1*}(a_n) \alpha^{k_n+k} \gamma_{2*}(b_n) \alpha^{l_n-k}.$$

Let  $m \in P$ , and consider the expression  $m = a_1 \alpha^{k_1} b_1 \alpha^{l_1} \dots a_n \alpha^{k_n} b_n \alpha^{l_n}$  chosen before. We then have

$$\varphi \circ \gamma_{k*}(m) = \varphi \circ \gamma_{1*}(a_1) \varphi(\alpha)^{k_1+k} \varphi \circ \gamma_{2*}(b_1) \varphi(\alpha)^{l_1-k} \dots \varphi \circ \gamma_{1*}(a_n) \varphi(\alpha)^{k_n+k} \varphi \circ \gamma_{2*}(b_n) \varphi(\alpha)^{l_n-k}.$$

Since  $\alpha a_i \alpha^{-1} a_i^{-1} \in P_1$ , we have  $\varphi \circ \gamma_{1*}(\alpha a_i \alpha^{-1} a_i^{-1}) \neq 1$ , but  $\gamma_{1*}(\alpha) = \alpha$ : hence  $\varphi \circ \gamma_{1*}(a_i)$  does not commute with  $\varphi(\alpha)$ . All the conditions of G. Baumslag's lemma are satisfied, and hence for all  $k$  large enough,  $\varphi \circ \gamma_{k*}$  sends every non trivial element of  $P$  on a non trivial element of  $\mathbb{F}$ .

Suppose finally that  $c_1$  is a non separating curve. Then  $\Sigma$ , this time, is obtained by glueing two boundary components of a surface  $\Sigma_1$ , and we have  $-\chi(\Sigma) = -\chi(\Sigma_1)$  but  $g(\Sigma_1) < g(\Sigma)$ . We have  $\pi_1 \Sigma = \langle \pi_1 \Sigma_1, t | t^{-1} \alpha t = \beta \rangle$ , where  $\alpha, \beta \in \pi_1 \Sigma_1$  are represented by the boundary components of  $\Sigma_1$  concerned by the glueing, and where  $t$  is represented, in  $\Sigma$ , by a simple curve intersecting the curve  $c_1$  at a single point. The elements  $m \in \pi_1 \Sigma$  can be written (non uniquely) as  $m = t^{k_0} a_1 t^{k_1} \dots a_n t^{k_n}$  with  $a_i \in \pi_1 \Sigma_1$ ; if  $\gamma$  is a diffeomorphism of  $\Sigma_1$  fixing its boundary (as well as a neighbourhood of  $c_1$  containing the base point) then  $\gamma_*(m) = t^{k_0} \gamma_*(a_1) t^{k_1} \dots \gamma_*(a_n) t^{k_n}$ , and the image of  $m$  under  $k$  Dehn twists along  $c_1$  is equal to  $(\alpha^k t)^{k_0} a_1 (\alpha^k t)^{k_1} \dots a_n (\alpha^k t)^{k_n}$ . The same argument as in the preceding case transposes here, thereby finishing the induction.  $\square$

### 3 Compactifications, degenerations and orientation

Before going into the study of the compactifications  $\overline{m}_g^u$  and  $\overline{m}_g^o$ , we will need to prove a technical fact, namely that the connected components of the spaces  $m_g^o$



and  $m_g^u$  are one-ended. This implies that for every compactification considered, the boundaries of these connected components are connected spaces.

### 3.1 The connected components of $m_g^u$ and $m_g^o$ are one-ended

Let us begin with some recalls.

**Definition 3.1** *Let  $X$  be a connected locally compact space. The supremum of the number of unbounded components (i.e., whose closure is non compact) of  $X \setminus K$ , as  $K$  describes the set of compact subsets of  $X$ , is called the number of ends of  $X$ .*

Here, once again, we are interested in the case  $\Gamma = \pi_1 \Sigma_g$  and  $n = 2$ ; we denote  $m_g^o = m_{\pi_1 \Sigma_g}^o(2)$ , with the same notations as before. We fix the generating set

$$S = \{a_1, a_1^{-1}, b_1, b_1^{-1}, \dots, b_g, b_g^{-1}\}.$$

Recall (corollary 2.14) that the function  $d: m_g^o \rightarrow \mathbb{R}_+$  defined as

$$d(\rho) = \min_{x \in \mathbb{H}^2} \max_{s \in S} d(x, \rho(s)x)$$

is well defined, continuous and proper. In particular  $m_g^o$  is locally compact, and, similarly,  $m_g^u$  is locally compact.

We are going to prove that the connected components of  $m_g^o$ , as well as those of  $m_g^u$ , are one-ended. Since the proof is exactly the same in both cases, until the end of this section (and *only* in this section) we will denote by  $m_g$  these representation spaces, and by  $m_{g,k}$  the corresponding connected components, being vague whether we consider the oriented or the unoriented representation space.

In [Hit87], N. Hitchin proved that for all  $k \in \{0, \dots, 2g-2\}$ , the connected component  $m_{g,k}$  of  $m_g$  is homeomorphic to a complex vector bundle of dimension  $g-1+|k|$  on the  $(2g-2-|k|)$ -th symmetric product of the surface. It follows, in particular, that the connected component  $m_{g,k}$  is one-ended, for every  $k \neq 0$ . We shall give a much more elementary proof (than the one of [Hit87]) of this result, and generalize it to the case  $k = 0$ .

**Proposition 3.2** *For all  $k$  such that  $|k| \leq 2g-2$ , the space  $m_{g,k}$  is one-ended.*

First let us fix some notations. If  $S' \subset S$ , we denote

$$d_{S'}(\rho) = \inf_{x \in \mathbb{H}^2} \max_{s \in S'} d(x, \rho(s)x).$$

If  $r > 0$  and  $S' \subset S$ , we denote

$$K_{S'}^r = \{[\rho] \subset m_g^o \mid d_{S'}(\rho) \leq r\},$$

$K^r = K_S^r$ ,  $U_{S'}^r = m_g^o \setminus K_{S'}^r$ . If  $s \in S$ ,  $U_{\{s\}}^r$  is therefore the set of conjugacy classes of representations  $\rho$  such that  $\rho(s)$  is a hyperbolic element whose translation length is strictly greater than  $r$ , which is equivalent to

$\text{Tr}(\rho(s)) > 2 \cosh r$ . We denote  $V^r = U_{\{a_1\}}^r \cap U_{\{a_2\}}^r$  et  $W^r = U_{\{a_1\}}^r \cup U_{\{a_2\}}^r$ . Of course,  $V^r \subset W^r \subset m_g^o \setminus K^r$ . Also, for all  $A \in PSL(2, \mathbb{R})$ , we wish to choose a one-parameter subgroup passing through  $A$ . If  $A = \text{Id}$ , we set  $A_t = \text{Id}$ , for all  $t \in \mathbb{R}$ . If  $A \in \text{Par} \cup \text{Hyp}$ , we choose  $A_t$  such that  $A_0 = \text{Id}$  and  $A_1 = A$ . If  $A \in \text{Ell}$ , we further require that  $A_t \neq \text{Id}$  for all  $t \in (0, 1]$ , and that  $A_t$  is a rotation of positive angle for small  $t$ . Note that for all  $n \in \mathbb{Z}$ , we have  $A_n = A^n$ .

We shall actually establish the following result:

**Proposition 3.3** *For every  $k \in \mathbb{Z}$  such that  $|k| \leq 2g - 2$ , and all  $r > 0$ , every two representations  $\rho_1, \rho_2 \in m_{g,k} \setminus K^{r+8g\delta_{\mathbb{H}^2}}$  can be joined by a path in  $m_{g,k} \setminus K^r$ .*

**Lemma 3.4** *Let  $A, B \in PSL(2, \mathbb{R})$  be such that  $\text{Tr}(B) > 2$ . Then for all  $r > 0$ , there exist  $n \in \mathbb{Z}$  and  $x \in \mathbb{R}$  such that  $\text{Tr}(A \cdot (BA_x)^n) > 2 \cosh r$ .*

**Proof**

We have  $B \in \text{Hyp}$ , so we can write  $B = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$  with  $\lambda > 1$ , and  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in some adapted basis. Then  $\text{Tr}(A \cdot B^n) = |a\lambda^n + \frac{d}{\lambda^n}|$ . This trace can be made arbitrarily large, as one takes  $|n|$  big enough, except in the case when  $a = d = 0$ , or, equivalently, when  $A$  is an elliptic element which exchanges the fixed points of  $B$  in  $\partial_{\infty}\mathbb{H}^2$ . In that case, if  $x$  is small enough, then  $B \cdot A_x$  is still a hyperbolic element, whose fixed points are distinct to those of  $B$  in  $\partial_{\infty}\mathbb{H}^2$  (indeed, up to conjugating  $\rho$  by a diagonal matrix, we have  $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  hence  $A_x = \begin{pmatrix} \cos \frac{x}{\pi} & -\sin \frac{x}{\pi} \\ \sin \frac{x}{\pi} & \cos \frac{x}{\pi} \end{pmatrix}$  and then  $B \cdot A_x = \begin{pmatrix} \lambda \cos \frac{x}{\pi} & -\lambda \sin \frac{x}{\pi} \\ \frac{1}{\lambda} \sin \frac{x}{\pi} & \frac{1}{\lambda} \cos \frac{x}{\pi} \end{pmatrix}$  does not fix 0 either  $\infty$  when  $\sin \frac{x}{\pi} \neq 0$ ).  $\square$

**Lemma 3.5** *Let  $A, B \in PSL(2, \mathbb{R})$  such that  $[A, B] \neq 1$ . Then there exist  $x \in \mathbb{R}$  and  $n \in \mathbb{Z}$  such that  $\text{Tr}(B \cdot (AB_x)^n) > 2$ .*

**Proof**

If  $B \in \text{Hyp}$ , we take  $n = 0$ . If  $A \in \text{Hyp}$ , then we exchange the roles of  $A$  and  $B$ , we apply lemma 3.4. If  $A$  or  $B$  is parabolic, and  $A, B \notin \text{Hyp}$ , then  $\text{Tr}(B \cdot (AB_x)^n) > 2$  as soon as  $x$  or  $n$  is large enough, as in lemma 3.4 (it suffices to write the matrices). Since  $A, B \neq \text{Id}$ , the only remaining case is when  $A, B \in \text{Ell}$ . For some small  $x$ , the matrix  $A \cdot B_x$  is elliptic, of infinite order (indeed,  $AB_x$  is still elliptic since traces are continuous, and one proves very easily (see e.g. [FW07], lemma 3.2) that this trace is non constant, so that the elliptic element  $AB_x$  has irrational angle for infinitely many  $x$ ), and does not commute with  $B$ . In some adapted basis,  $B = \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}$ , and  $AB_x = U^{-1} \cdot \begin{pmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{pmatrix} \cdot U$ , where  $U$  sends the fixed point of  $B$

on that of  $AB_x$ . In particular, we can require that  $U$  is hyperbolic. Since  $\theta_2$  is irrational,  $(AB_x)^n$  can be arbitrarily close to  $U^{-1}B^{-1}U$ , and  $\text{Tr}(BU^{-1}B^{-1}U)$  is the trace of the commutator of a hyperbolic element with an elliptic element in  $PSL(2, \mathbb{R})$ , and we can check by easy computation that this is always strictly greater than 2. Therefore,  $\text{Tr}(B \cdot (AB_x)^n) > 2$ , for some  $n \in \mathbb{Z}$ .  $\square$

**Lemma 3.6** *Let  $\rho \in m_g \setminus K^{r+8g\delta_{\mathbb{H}^2}}$ . Then there exist  $s_1, s_2 \in \{a_1, \dots, b_g\}$  such that  $d_{\{s_1, s_2\}}(\rho) > r$ .*

**Proof**

If  $\rho \in R_g$ ,  $S' \subset S$  and  $\alpha > 0$  denote  $F_{S'}^\alpha(\rho) = \{x \in \mathbb{H}^2 \mid \forall s \in S', d(x, \rho(s)x) \leq \alpha\}$ .

Let  $\alpha > 0$ , and suppose that  $F_{\{s_1, s_2\}}^\alpha(\rho) \neq \emptyset$  for every pair  $\{s_1, s_2\} \subset \{a_1, \dots, b_g\}$ . Then for every triple  $\{s_1, s_2, s_3\}$ , the convex sets  $F_{\{s_1\}}^\alpha(\rho)$ ,  $F_{\{s_2\}}^\alpha(\rho)$ ,  $F_{\{s_3\}}^\alpha(\rho)$  intersect pairwise, hence, since  $\mathbb{H}^2$  is  $\delta_{\mathbb{H}^2}$ -hyperbolic, there exists a point  $x$  at distance at most  $2\delta_{\mathbb{H}^2}$  of each of these three sets. This implies that  $x \in F_{\{s_1, s_2, s_3\}}^{\alpha+4\delta_{\mathbb{H}^2}}(\rho)$ . More generally, if  $x > 0$  is such that for every  $k$ -tuple  $S'$  of  $\{a_1, \dots, b_g\}$ ,  $F_{S'}^x \neq \emptyset$ , then, if  $\{s_1, \dots, s_{k+1}\} \subset \{a_1, \dots, b_g\}$ , the convex sets  $F_{\{s_1, \dots, s_{k-1}\}}^x(\rho)$ ,  $F_{\{s_1, \dots, s_{k-2}, s_k\}}^x(\rho)$  and  $F_{\{s_1, \dots, s_{k-2}, s_{k+1}\}}^x(\rho)$  intersect pairwise, and it follows that  $F_{\{s_1, \dots, s_{k+1}\}}^{x+4\delta_{\mathbb{H}^2}}(\rho) \neq \emptyset$ . Therefore, by induction on  $\text{Card}(\{a_1, \dots, b_g\})$  we get that  $d(\rho) \leq \alpha + 8g\delta_{\mathbb{H}^2}$ , as soon as  $F_{\{s_1, s_2\}}^\alpha(\rho) \neq \emptyset$  for every pair  $\{s_1, s_2\} \subset \{a_1, \dots, b_g\}$ .

Since the inequality  $d(\rho) > r + 8g\delta_{\mathbb{H}^2}$  is strict, there exists  $\varepsilon > 0$  such that  $d(\rho) > r + 8g\delta_{\mathbb{H}^2} + \varepsilon$ , which implies that there exists a pair  $\{s_1, s_2\} \subset \{a_1, \dots, b_g\}$  such that  $F_{\{s_1, s_2\}}^{r+\varepsilon}(\rho) = \emptyset$ ; this implies that  $d_{\{s_1, s_2\}}(\rho) \geq r + \varepsilon$  hence  $d_{\{s_1, s_2\}}(\rho) > r$ .  $\square$

**Proof** of proposition 3.3.

**Step 1:** *Let  $\rho_0, \rho_1 \in V^r \cap e^{-1}(k)$ . Then there exists a path  $\rho_t \in W^r$ , joining  $\rho_0$  and  $\rho_1$ .*

By lemma 10.1 of W. Goldman [Gol88], for every  $k \in \mathbb{Z}$  such that  $|k| \leq 2g-2$ , the set of representations  $\rho$  such that  $[\rho(a_i), \rho(b_i)] \neq \text{Id}$  is path-connected and dense in  $m_{g,k}$ . We can thus perturb  $\rho_0$  and  $\rho_1$ , and find a path  $\rho_t \in m_{g,k}$  joining  $\rho_0$  to  $\rho_1$  and such that for all  $t \in [0, 1]$  and all  $i \in \{1, \dots, g\}$ ,  $[\rho_t(a_i), \rho_t(b_i)] \neq \text{Id}$ .

Let  $t \in [0, 1]$  and  $i \in \{1, 2\}$ . Denote  $A(t) = \rho_t(a_i)$  and  $B(t) = \rho_t(b_i)$ . Then by lemma 3.5, there exist  $n_i(t) \in \mathbb{Z}$  and  $x_i(t) \in \mathbb{R}$  such that  $\text{Tr}\left(B(t)(A(t)B(t)_{x_i(t)})^{n_i(t)}\right) > 2$ , and, of course, we have

$$\left[A(t)B(t)_{x_i(t)}, B(t)(A(t)B(t)_{x_i(t)})^{n_i(t)}\right] = [A(t), B(t)].$$

Hence, by lemma 3.4, there exist  $n'_i(t) \in \mathbb{Z}$  and  $x'_i(t) \in \mathbb{R}$  such that

$$\text{Tr}\left(A(t)B(t)_{x_i(t)}\left(B(t)(A(t)B(t)_{x_i(t)})^{n_i(t)}(A(t)B(t)_{x_i(t)})_{x'_i(t)}\right)^{n'_i(t)}\right) > 2 \cosh r, \quad (3)$$

and we have again

$$\left[ AB_{x_i(t)} \cdot \left( B(AB_{x_i(t)})^{n_i(t)} \cdot (AB_{x_i(t)})_{x'_i(t)} \right)^{n'_i(t)}, B(AB_{x_i(t)})^{n_i(t)} \right] = [A, B]. \quad (4)$$

The equality (3) being strict, for all  $\tau \in [0, 1]$  there exists an interval  $(\tau - \delta, \tau + \delta)$  such that for all  $t \in (\tau - \delta, \tau + \delta) \cap [0, 1]$  we have

$$\text{Tr} \left( A(t)B(t)_{x_i(\tau)} \left( B(t)(A(t)B(t)_{x_i(\tau)})^{n_i(\tau)} (A(t)B(t)_{x_i(\tau)})_{x'_i(\tau)} \right)^{n'_i(\tau)} \right) > 2 \cosh r.$$

The compact set  $[0, 1]$  is covered by finitely many such intervals, hence there exists a subdivision  $0 = t_0 < t_1 < \dots < t_k = 1$ , and elements  $n_i^j, n_i^{j'} \in \mathbb{Z}$ ,  $x_i^j, x_i^{j'} \in \mathbb{R}$ , such that for every  $i \in \{1, 2\}$  and  $j \in \{0, \dots, k-1\}$  such that for all  $t \in [t_j, t_{j+1}]$  we have

$$\text{Tr} \left( A(t)B(t)_{x_i^j} \left( B(t)(A(t)B(t)_{x_i^j})^{n_i^j} (A(t)B(t)_{x_i^j})_{x_i^{j'}} \right)^{n_i^{j'}} \right) > 2 \cosh r, \quad (5)$$

with  $x_i^0 = x_i^{0'} = n_i^0 = n_i^{0'} = 0$ . We also set  $x_i^k = x_i^{k'} = n_i^k = n_i^{k'} = 0$ . Fix  $j \in \{0, \dots, k-1\}$ . For all  $t \in [t_j, t_{j+1}]$  and  $i \in \{1, 2\}$ , we set

$$f_t^j(a_i) = A_i(t)B_i(t)_{x_i^j} \cdot \left( B_i(t)(A_i(t)B_i(t)_{x_i^j})^{n_i^j} \cdot (A_i(t)B_i(t)_{x_i^j})_{x_i^{j'}} \right)^{n_i^{j'}}$$

and

$$f_t^j(b_i) = B_i(t)(A_i(t)B_i(t)_{x_i^j})^{n_i^j},$$

with  $A_i(t) = \rho_t(a_i)$  and  $B_i(t) = \rho_t(b_i)$ , and we put  $f_t^j(a_i) = \rho_t(a_i)$ ,  $f_t^j(b_i) = \rho_t(b_i)$  for  $i \leq 3$ . The identity (4) guarantees that we are indeed defining a representation  $f_t^j \in R_g$ , for all  $j \in \{0, \dots, k-1\}$  and  $t \in [t_j, t_{j+1}]$ . Note that we have  $f_0^0 = \rho_0$ . Denote also  $f_1^k = \rho_1$ .

Now we can construct a path joining  $\rho_0$  to  $\rho_1$  in  $W^r$ , as follows:

- The path  $f_t^j$  enables to travel from  $f_{t_j}^j$  to  $f_{t_{j+1}}^j$  without leaving  $W^r$ , by (5).
- In order to go from  $f_{t_{j+1}}^j$  to  $f_{t_{j+1}}^{j+1}$ , we first use Dehn twists in the first handle, and then we use Dehn twists in the second handle. More precisely, for every  $j$  between 0 and  $k-1$ , denote  $y_i(t) = tx_i^j + (1-t)x_i^{j+1}$ ,  $y'_i(t) = tx_i^{j'} + (1-t)x_i^{j'+1}$ ,  $m_i(t) = tn_i^j + (1-t)n_i^{j+1}$ ,  $m'_i(t) = tn_i^{j'} + (1-t)n_i^{j'+1}$ , et  $A_i = A_i(t_{j+1})$ ,  $B_i = B_i(t_{j+1})$ . We then set

$$g_t^j(a_1) = A_1 B_{1y_1(t)} \cdot \left( B_1(A_1 B_{1y_1(t)})_{m_1(t)} \cdot (A_1 B_{1y_1(t)})_{y'_1(t)} \right)_{m'_1(t)},$$

$$g_t^j(b_1) = B_1(A_1 B_{1y_1(t)})_{m_1(t)}$$

and  $g_t^j(x) = f_{t_{j+1}}^j(x)$  for all  $t \in [0, 1]$  and all  $x \in \{a_2, b_2, \dots, a_g, b_g\}$ , and we set

$$h_t^j(a_2) = A_2 B_{2y_2(t)} \cdot \left( B_2(A_2 B_{2y_2(t)})_{m_2(t)} \cdot (A_2 B_{2y_2(t)})_{y'_2(t)} \right)_{m'_2(t)},$$

$$h_t^j(b_2) = B_2(A_2 B_{2y_2(t)})_{m_2(t)}$$

and  $h_t^j(x) = f_{t_{j+1}}^{j+1}(x)$  for all  $t \in [0, 1]$  and all  $x \in \{a_1, b_1, a_3, b_3, \dots, a_g, b_g\}$ .

We then have  $g_0^j = f_{t_{j+1}}^j$ ,  $g_1^j = h_0^j$  and  $h_1^j = f_{t_{j+1}}^{j+1}$ , and the equality (5) still guarantees that these paths do not leave  $W^r$ .

**Step 2:** Let  $\rho \in m_{g,k} \setminus K^{r+8g\delta_{\mathbb{H}^2}}$ . Then there exists a path  $\rho_t$  taking values in  $m_{g,k} \setminus K^r$ , such that  $\rho_0 = \rho$  and  $\rho_1 \in V^r$ .

By lemma 3.6, there exist  $s_1, s_2 \in \{a_1, \dots, b_g\}$  such that  $F_{\{s_1, s_2\}}^r(\rho) \neq \emptyset$ . If  $g \geq 3$ , then there exists  $i \in \{1, \dots, g\}$  such that  $\{a_i, b_i\} \cap \{s_1, s_2\} = \emptyset$ . In that case, as in step 1, we can use Dehn twists in the handle  $i$ , without entering  $K^r$  since we do not touch  $\rho(s_1)$ ,  $\rho(s_2)$ . This completes the proof of proposition 3.3, in the case  $g \geq 3$ .

In the case when  $g = 2$  and  $\{s_1, s_2\} = \{a_1, b_1\}$  or  $\{s_1, s_2\} = \{a_2, b_2\}$ , we do the same as in the preceding cases. Now suppose for instance that  $s_1 = a_1$  et  $s_2 = a_2$  (the other cases are dealt with similarly). If  $\rho(a_1)$  or  $\rho(a_2)$  is hyperbolic (say, for instance,  $\rho(a_1)$ ), then as in lemma 3.4, Dehn twists enable to get  $\text{Tr}(\rho_t(b_1)) > 2 \cosh r$  (almost) without touching  $\rho(a_1)$  and  $\rho(a_2)$ . If  $\rho(a_1)$  or  $\rho(a_2)$  is parabolic (say,  $\rho(a_1)$ ), then one proves easily (see e.g. lemma 3.2) that  $\text{Tr}(\rho(a_1)\rho(b_1)_t)$  is non constant at  $t = 0$ , and in that way we can get to the case when  $\rho(a_1)$  is hyperbolic.

The last case, in which  $\rho(a_1), \rho(a_2) \in Ell$ , is still going to require some work. Up to conjugating  $\rho$ , we have  $\rho(a_1) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ , and  $\rho(b_1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . There exists  $\alpha \in \mathbb{R}$  such that the vectors  $\begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$  and  $\begin{pmatrix} a+d \\ b-c \end{pmatrix}$  of  $\mathbb{R}^2$  are colinear. And  $(a+d)^2 + (b-c)^2 = 4 + (a-d)^2 + (b+c)^2 > 4$ , since  $\rho(b_1)$  does not commute with  $\rho(a_1)$  (indeed, if  $a = d$  and  $b = -c$  then  $a^2 + b^2 = 1$  hence  $\rho(b_1)$  is under the form  $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$ ). We then have  $\text{Tr} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \right) > 2$ , so that, for some  $t \in \mathbb{R}$ ,  $\rho(b_1)\rho(a_1)_t \in Hyp$ . Note also that the axis of that hyperbolic element passes through the fixed point of  $\rho(a_1)$  (indeed,  $\rho(b_1)\rho(a_1)_t$  is of the form  $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$ ). By conjugating this element by the rotation  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  we get  $(\rho(b_1)\rho(a_1)_t)^{-1}$ , so the element  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  exchanges the fixed points of  $\rho(b_1)\rho(a_1)_t$ , which implies that the axis of  $\rho(b_1)\rho(a_1)_t$  passes through the fixed

point of  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ). This is not extremely important for the proof, but explains figure 1.

Denote  $M = [\rho(a_1), \rho(b_1)]$ . It is the commutator of two non-commuting elliptic elements in  $PSL(2, \mathbb{R})$ . An easy computation then yields  $\text{Tr}(M) > 2$ , hence  $M \in Hyp$ . The number  $d_{\{a_1, a_2\}}(\rho)$  depends only on the angles of the rotations  $\rho(a_1)$  and  $\rho(a_2)$ , and of the distance between their respective centres  $x_1, x_2 \in \mathbb{H}^2$ . Regardless of the repartition of these fixed points and the axis of  $M$ , there exists  $\varepsilon \in \{-1, 1\}$  such that for all  $t > 0$ ,  $d(x_1, M_{\varepsilon t} \cdot x_2) > d(x_1, x_2)$ . Denote by  $p \in \partial \mathbb{H}^2$  the attractive point of  $M_\varepsilon$ . By conjugating  $\rho(a_2)$  and  $\rho(b_2)$  by  $M_{\varepsilon t}$  (equivalently, by making a Dehn twists along the simple closed curve representing  $[a_1, b_1]$ ), we get a representation  $\rho_t$  such that  $d_{\{a_1, a_2\}}(\rho_t)$  is arbitrarily large; the fixed point of  $\rho_t(a_2)$  being sent to  $p$ .

Now suppose that  $\rho(b_1)$  is hyperbolic, with its axis passing through the fixed point of  $\rho(a_1)$ . Then for all  $t$  such that  $|t|$  is small enough, the element  $\rho(a_1)\rho(b_1)_t$  is elliptic; its fixed point can be found as suggests the following picture:

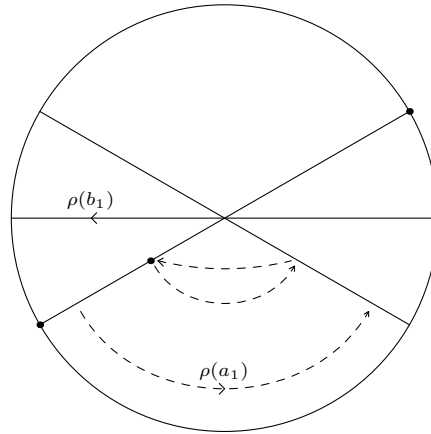


Figure 1: The element  $\rho(a_1) \cdot \rho(b_1)_t$ .

There exist two real numbers  $t_1$  and  $t_2$  such that  $\rho(a_1)\rho(b_1)_{t_i}$  are parabolic elements, whose unique fixed point is one of the points drawn on the above picture. At least one of these points does not depend on  $p$ ; denote by  $T = t_i$  a corresponding element. Then for all  $t \in [0, T]$ ,  $p$  is not a fixed point of  $\rho(a_1)\rho(b_1)_t$ . Hence, if we have sent the fixed point of  $\rho(a_2)$  far enough, this defines a path  $\rho'_t$  such that  $d_{\{a_1, a_2\}}(\rho'_t)$  stays large, and for all  $t$  big enough,  $\rho'_t(a_1) \in Par$ , hence we recover the preceding situation.  $\square$

Similarly, we can prove that every loop can be pushed out of every compact set. In other words, the fundamental group of the space  $e^{-1}(k)$  is entirely carried by this only end.

This fact will be useful here for the following reason:

**Corollary 3.7** *Let  $k \in \{2 - 2g, \dots, 2g - 2\}$  and denote  $X = m_{g,k}$ . Let  $(\overline{X}, i)$  be a compactification of  $X$ . Then  $\partial\overline{X}$  is connected.*

**Proof**

The space  $X$  has at least one end, hence is non compact: therefore,  $\partial\overline{X}$  is non empty. Suppose that  $\partial\overline{X} = A \cup B$ , where  $A$  and  $B$  are disjoint, open and closed subsets of  $\partial\overline{X}$ . The boundary  $\partial\overline{X}$  being closed,  $A$  and  $B$  are closed subsets of  $\overline{X}$ . Since  $\overline{X}$  is compact Hausdorff, it is normal. Hence, there exist two open subsets  $U$  and  $V$  of  $\overline{X}$  such that  $A \subset U$ ,  $B \subset V$ , and  $U \cap V = \emptyset$ . The open set  $U \cup V$  contains  $\partial\overline{X}$  hence it is the complement of a compact subset  $K$  of  $X$ . This compact set  $K$  is contained in  $K^r$ , for  $r$  large enough. And  $X \setminus K^r$  possesses a unique unbounded connected component, denote it by  $W$ . By proposition 3.3,  $X \setminus W$  is contained in  $K^{r+8g\delta_{\mathbb{H}^2}}$ , thus  $K' = X \setminus W$  is a compact subset of  $X$ . Now  $W$  is dense in  $W' = \overline{X} \setminus K'$ , which is therefore connected. Denote  $U' = W' \cap U$  and  $V' = W' \cap V$ . Then  $U'$  contains  $A$ ,  $V'$  contains  $B$ ,  $U'$  and  $V'$  are open,  $U' \cap V' = \emptyset$ , and  $W' = U' \cup V'$  is connected. This implies that  $U' = \emptyset$  or  $V' = \emptyset$ , hence  $A = \emptyset$  or  $B = \emptyset$ .  $\square$

## 3.2 Oriented compactification

In most of the sequel we will consider the case  $n = 2$ , and we consider the compactification of the space  $m_\Gamma^u$  of actions on the hyperbolic plane  $\mathbb{H}^2$ . We will prove, at least in the case when  $\Gamma$  is a surface group, that this space is degenerated; and it seems more natural to study a compactification of the oriented version  $m_\Gamma^o$ .

As we said before, the ideal points of the compactification  $\overline{m_g^u}$  of M. Bestvina and F. Paulin are (equivariant isometry classes of) actions of  $\pi_1\Sigma_g$  by isometries on  $\mathbb{R}$ -trees. We shall prove that it is possible to equip these  $\mathbb{R}$ -trees with an orientation, so that we can define an Euler class on these trees. This will enable to define a compactification of  $m_g^o$ , in which the Euler class extends continuously to the boundary.

### 3.2.1 Fat $\mathbb{R}$ -trees

Let  $X$  be a hyperbolic space in the sense of Gromov. A *germ of rays*, in  $X$ , is an equivalence class of rays, for the following equivalence relation: we say that two rays are equivalent if they coincide on some non trivial initial segment.

In the rest of this section,  $T$  will be an  $\mathbb{R}$ -tree non reduced to a point. At every point  $x \in T$ , denote by  $\mathcal{G}(x)$  the set of germs of rays issued of  $x$ . An *orientation of  $T$*  is the data, for all  $x \in T$ , of a total cyclic order  $or(x)$  in  $\mathcal{G}(x)$ .

### Definition 3.8

- An  $\mathbb{R}$ -tree equipped with an orientation is called a *fat  $\mathbb{R}$ -tree*.
- Let  $(T, or)$  and  $(T', or')$  be two fat  $\mathbb{R}$ -trees and let  $h \in \text{Isom}(T, T')$  be an isometry. Of course,  $h$  defines, at every point  $x \in T$ , a bijection

$\mathcal{G}h_x: \mathcal{G}(x) \rightarrow \mathcal{G}(h(x))$ . We say that  $h$  preserves the orientation if for every  $x \in T$ , the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}(x)^3 & \xrightarrow{or(x)} & \{-1, 0, 1\} \\ (\mathcal{G}h_x)^3 \downarrow & \nearrow or'(h(x)) & \\ \mathcal{G}(h(x))^3 & & \end{array}$$

The set of isometries of  $T$  which preserve the orientation or form a subgroup of  $Isom(T)$ , which will denote  $Isom^{or}(T)$ .

- We say that an action  $\rho: \Gamma \rightarrow Isom(X)$  of a group  $\Gamma$  by isometries preserves the orientation or if it takes values into  $Isom^{or}(T)$ .

Since we are interested in defining the Euler class of actions on  $\mathbb{R}$ -trees preserving the orientation, we need to consider more particularly the boundary of the tree. We say that a total cyclic order  $o$  on  $\partial_\infty T$  is *coherent* if for every  $x \in T$  and every non degenerate triple  $\{c_1, c_2, c_3\}$  of rays issued from  $x$ , the element  $o(r_1, r_2, r_3)$  does not depend on the chosen representants  $r_1, r_2, r_3$  of  $c_1, c_2, c_3$ .

For instance, in the following configuration



a total cyclic order  $o$  on the boundary  $\{r_1, r_2, r_3, r_4\}$  is coherent if and only if it verifies  $o(r_1, r_2, r_3) = o(r_1, r_2, r_4)$  et  $o(r_1, r_3, r_4) = o(r_2, r_3, r_4)$ .

Of course, there is a relation between orientations on an  $\mathbb{R}$ -tree and coherent total cyclic orders on its boundary. Let us begin with some notations.

Naturally, we say that an  $\mathbb{R}$ -tree  $T$  is the *convex hull of its boundary* if it is the closure of the union of the geodesics which are isometric to the whole  $\mathbb{R}$ . In other words, if

$$T = \overline{\bigcup_{\substack{a, b \in \partial_\infty T \\ a \neq b}} (a, b)}.$$

We also say that two non degenerate tripods are equivalent if their intersection is a non degenerate tripod; we denote by  $Trip(T)$  the set of equivalence classes. Denote

$$O'_T = \{-1, 1\}^{Trip(T)},$$

and equip it with the product topology (the topology of pointwise convergence). By Tikhonov's theorem, it is a compact space. Denote by  $O_T$  the (closed) set of functions  $o$  which satisfy the following conditions: for every tripod  $Trip(a_0, a_1, a_2, a_3)$  (where  $a_0$  is the central point), we have

$$o(Trip(a_0, a_1, a_2, a_3)) = o(Trip(a_0, a_2, a_3, a_1)) = -o(Trip(a_0, a_1, a_3, a_2)).$$



Non degenerate triples of germs of rays issued from a same point are naturally identified with classes of tripods, so that an orientation of  $T$  is naturally identified with an element of  $O_T$ . More precisely, denote by  $OR_T \subset O_T$  the set consisting of functions satisfying the following condition: for all  $a \in T$  and all tripods  $Trip(a, b, c, d)$  and  $Trip(a, b, d, e)$  of central point  $a$ ,

$$or(Trip(a, b, c, d)) = or(Trip(a, b, d, e)) = 1 \Rightarrow or(Trip(a, b, c, e)) = 1.$$

Then  $OR_T$  identifies to the set of orientations of  $T$ ; remark that  $OR_T$  is a closed set.

Now let  $E_T$  be the set

$$E_T = \{-1, 0, 1\}^{(\partial T)^3},$$

equipped with the product topology. Let  $E'_T$  be the (closed) subset of  $E_T$ , consisting of elements  $o \in E_T$  satisfying:

$$\forall (x, y, z) \in (\partial T)^3, \quad o(x, y, z) = 0 \Leftrightarrow Card\{x, y, z\} \leq 2.$$

Denote by  $ORD_T$  the set of coherent cyclic orders on  $\partial_\infty T$ .

For every non degenerate triple  $(x, y, z) \in (\partial_\infty T)^3$ , denote by  $t(x, y, z) \in Trip(T)$  the class of the tripod  $(x, y) \cup (x, z) \cup (y, z)$ . This function  $t$  defines, by precomposition, a function  $f: O_T \rightarrow E_T$ .

**Proposition 3.9** *Suppose that  $T$  is the convex hull of its boundary. Then the restriction of  $f$  to  $OR_T$  is a homeomorphism on its image  $ORD_T$ .*

**Proof**

Denote

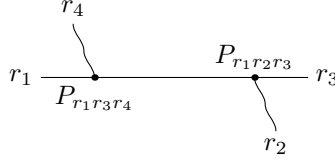
$$T' = \bigcup_{\substack{a, b \in \partial_\infty T \\ a \neq b}} (a, b).$$

We easily check that  $T'$  is connected (if  $x, y \in T'$ , then  $x \in (a, b)$  and  $y \in (c, d)$ : then  $(a, c) \cap (a, b)$  is non empty since  $T$  is a tree, thus there exists a path in  $T'$  joining  $x$  to a point of  $(a, c)$ , and  $(a, c) \cap (c, d) \neq \emptyset$  guarantees that there is a path in  $T'$  joining  $x$  to a point of  $(c, d)$ , and then to  $y$ ), hence  $T'$  is a subtree of  $T$ , dense in  $T$ . Therefore, every class of tripods possesses a representant in  $T'$ ; one checks also that every segment in  $T'$  is a subset of some line  $(a, b)$ . The function  $t$  is thus surjective, so that  $f$  is injective.

The continuity of  $f$  is straightforward, and hence  $f$  is a homeomorphism from  $OR_T$  (which is compact) on its image (which is Hausdorff); now we need to check that  $f(OR_T) = ORD_T$ .

Let  $or$  be an orientation of  $T$ , and  $o = f(or)$ . It is immediate that the function  $o: \partial_\infty T^3 \rightarrow \{-1, 0, 1\}$  satisfies the conditions 1 and 2 of definition 2.20. Let us check that condition 3 is also satisfied: suppose that  $o(r_1, r_2, r_3) =$

$o(r_1, r_3, r_4) = 1$ . For every non degenerate triple  $(a, b, c) \in \partial_\infty T$ ,  $(a, b) \cap (a, c) \cap (b, c)$  is a singleton; denote it by  $\{P_{abc}\}$ . If  $P_{r_1 r_3 r_4} \in (r_1, P_{r_1 r_2 r_3})$ , then



$P_{r_1 r_2 r_4} = P_{r_1 r_3 r_4}$  and the germs of  $[P_{r_1 r_2 r_4}, r_2)$  and  $[P_{r_1 r_2 r_4}, r_3)$  are identical, so that  $o(r_1, r_2, r_4) = 1$ . If  $P_{r_1 r_3 r_4} \in (P_{r_1 r_2 r_3}, r_3)$  the argument is the same. If  $P_{r_1 r_3 r_4} = P_{r_1 r_2 r_3}$ , then the germs of the rays  $[P_{r_1 r_2 r_3}, r_2)$  and  $[P_{r_1 r_2 r_3}, r_4)$  are distinct since we have  $o(r_1, r_3, r_2) \neq o(r_1, r_3, r_4)$ : hence  $[P_{r_1 r_2 r_3}, r_1), \dots, [P_{r_1 r_2 r_3}, r_4)$  define four germs of rays issued from  $P_{r_1 r_2 r_3}$ , which are all distinct, hence  $o(r_1, r_2, r_4) = 1$ , since  $or(P_{r_1 r_2 r_3})$  satisfies condition 3 of definition 2.20. And by construction, the order  $o$  is coherent. We have just proved that  $f(OR_T) \subset ORD_T$ . But by definition, a total cyclic order is coherent if and only if it is in the image of  $f$ , and if  $f(or)$  is a coherent total cyclic order we check easily that  $or$  is indeed an orientation of  $T$ .  $\square$

Now recall that an action of a group  $\Gamma$  by isometries on an  $\mathbb{R}$ -tree  $T$  is called *minimal* if  $T$  possesses no subtree  $T' \subset T$ , invariant under the action of  $\Gamma$ , distinct from  $\emptyset$  and  $T$ .

In the sequel,  $\Gamma$  is a finitely generated group and we consider minimal actions of  $\Gamma$  on  $\mathbb{R}$ -trees. In that case,  $T$  is the union of the translation axes of the hyperbolic elements in the image of  $\Gamma$  (see e.g. [MS84, Pau89]). In particular, such trees are the convex hulls of their boundaries.

In the sequel, we shall need to consider the *set* of classes of minimal actions of  $\Gamma$  on fat  $\mathbb{R}$ -trees, preserving the orientation, up to equivariant isometry preserving the order. General arguments of cardinality enable to do that, but the following proposition enables us to see this as an explicit set.

If  $(T, or)$  is a fat  $\mathbb{R}$ -tree and if  $u, v, w \in T$  are not aligned, then they define a class of tripods and we denote by  $or(u, v, w) \in \{-1, 1\}$  the image of this tripod by  $or$ . If  $u, v, w$  are aligned, then we write  $or(u, v, w) = 0$ .

**Proposition 3.10** *Let  $(T, or)$  be a fat  $\mathbb{R}$ -tree, let  $x_0 \in T$ , and let  $\rho: \Gamma \rightarrow Isom^{or}(T)$  be a minimal action of a finitely generated group  $\Gamma$ , preserving the orientation. Then the  $\mathbb{R}$ -tree  $T$ , the orientation  $or$  and the action  $\rho$  are entirely determined by the functions  $f: \Gamma^2 \rightarrow \mathbb{R}$  et  $g: \Gamma^3 \rightarrow \{-1, 0, 1\}$  defined by*

$$f(\gamma_1, \gamma_2) = d_T(\gamma_1 x_0, \gamma_2 x_0) \quad \text{et} \quad g(\gamma_1, \gamma_2, \gamma_3) = o(\gamma_1 x_0, \gamma_2 x_0, \gamma_3 x_0).$$

*More precisely, if  $\rho: \Gamma \rightarrow Isom^{or}(T)$  and  $\rho': \Gamma \rightarrow Isom^{or'}(T')$  define the same functions  $f$  and  $g$ , for some choices of base points in  $T$  and  $T'$ , then there exists an equivariant isometry  $\varphi: T \rightarrow T'$  preserving the order and the base point.*

### Proof

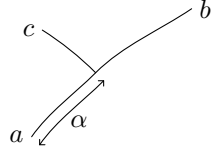
This is well-known (see e.g. [Pau89]) for minimal actions of a finitely generated groups on  $\mathbb{R}$ -trees; here we simply need to add the orientation.

Let us begin with the following remark. Let  $T$  and  $T'$  be two fat  $\mathbb{R}$ -trees and  $x_1, \dots, x_n \in T$ ,  $x'_1, \dots, x'_n \in T'$  such that for all  $i, j$ ,  $d(x_i, x_j) = d(x'_i, x'_j)$  and all  $i, j, k$ ,  $or(x_i, x_j, x_k) = or'(x'_i, x'_j, x'_k)$ . Denote  $K = \{x_1, \dots, x_n\}$ . Then the function  $\varphi_K: \{x_1, \dots, x_n\} \rightarrow \{x'_1, \dots, x'_n\}$  defined by  $\varphi_K(x_i) = x'_i$  extends uniquely to an orientation-preserving isometry

$$\varphi_{Hull(K)}: Hull(\{x_1, \dots, x_n\}) \rightarrow Hull(\{x'_1, \dots, x'_n\})$$

(the subsets  $Hull(\{x_1, \dots, x_n\})$  and  $Hull(\{x'_1, \dots, x'_n\})$ , as subtrees of  $T$  and  $T'$ , are oriented trees, by the restrictions of  $or$  and  $or'$ ).

We prove this by induction. If  $n = 1$ , there is not very much to do. Denote again  $K = \{x_1, \dots, x_n\}$  and suppose that  $\varphi_{Hull(K)}$  is an orientation-preserving isometry between  $Hull(\{x_1, \dots, x_n\})$  and  $Hull(\{x'_1, \dots, x'_n\})$ . Denote by  $y_{n+1}$  the projection of  $x_{n+1}$  on  $Hull(K)$ . The following relation, true in every  $\mathbb{R}$ -tree,



$$\alpha = \frac{1}{2} (d(a, b) + d(a, c) - d(b, c))$$

enables to find  $y_{n+1}$  in the tree  $Hull(K)$ : the real numbers  $d(x_1, x_{n+1}), \dots, d(x_n, x_{n+1})$  determine a unique point  $y_{n+1} \in Hull(K)$ ; and similarly they determine a unique point  $y'_{n+1} \in Hull(\{x'_1, \dots, x'_n\})$ , and we have  $\varphi_{Hull(K)}(y_{n+1}) = y'_{n+1}$ . If  $x_{n+1} = y_{n+1}$  then  $Hull(\{x_1, \dots, x_n\}) = Hull(\{x_1, \dots, x_{n+1}\})$  and  $Hull(\{x'_1, \dots, x'_n\}) = Hull(\{x'_1, \dots, x'_{n+1}\})$  and the induction is proved. Otherwise,  $Hull(\{x_1, \dots, x_{n+1}\})$  is obtained by glueing at the point  $y_{n+1}$  the tree  $Hull(\{x_1, \dots, x_n\})$  and the segment  $[x_{n+1}, y_{n+1}]$ , whose length is determined by the real numbers  $d(x_1, x_{n+1}), \dots, d(x_n, x_{n+1})$ ; and similarly for  $Hull(\{x'_1, \dots, x'_{n+1}\})$  in  $T'$ . In that way, the isometry  $\varphi_{Hull(K)}$  extends to a unique isometry  $\varphi_{Hull(\{x_1, \dots, x_{n+1}\})}$ . Since  $\varphi_{Hull(K)}$  preserves the orientation, we need only check that  $\varphi_{Hull(\{x_1, \dots, x_{n+1}\})}$  preserves the orientation at the vertex  $y_{n+1}$ . But this follows from the fact that all the classes of the tripods of centre  $y_{n+1}$  have a representant of the type  $Hull(\{x_i, x_j, x_{n+1}\})$ , where  $Card(\{i, j, n+1\}) = 3$ .

Now suppose that  $(\rho, T)$  and  $(\rho', T')$  define the same functions  $f$  and  $g$ ; denote by  $x_0$  and  $x'_0$  the base points. Let  $P$  be a finite subset of  $\Gamma$ . We then have a unique isometry  $\varphi_{Hull(P \cdot x_0)}$  between  $Hull(P \cdot x_0)$  and  $Hull(P \cdot x'_0)$ , preserving the orientation, such that  $\varphi_{Hull(P \cdot x_0)}(\gamma \cdot x_0) = \gamma x'_0$ . In particular, for every finite subset  $Q$  of  $P$  the restriction of  $\varphi_{Hull(P \cdot x_0)}$  to  $Hull(Q \cdot x_0)$  equals  $\varphi_{Hull(Q \cdot x_0)}$ , so that we can construct an isometry  $\varphi: \bigcup_{P \subset \Gamma} Hull(P \cdot x_0) \rightarrow \bigcup_{P \subset \Gamma} Hull(P \cdot x'_0)$ , such that for all  $\gamma \in \Gamma$  we have  $\varphi(\gamma \cdot x_0) = \gamma \cdot x'_0$ ; and we also deduce that for every  $\gamma \in \Gamma$  and every finite subset  $P$  of  $\Gamma$  we have

$$\forall y \in Hull(P \cdot x_0), \quad \rho'(\gamma) \cdot \varphi_{Hull(P \cdot x_0)} = \varphi_{Hull(\gamma P \cdot x_0)}(\rho(\gamma) \cdot y)$$

which ensures that the isometry  $\varphi$  is equivariant for the actions  $\rho, \rho'$  on the trees  $\bigcup_{P \subset \Gamma} \text{Hull}(P \cdot x_0)$  and  $\bigcup_{P \subset \Gamma} \text{Hull}(P \cdot x'_0)$ . Every tripod in  $\bigcup_{P \subset \Gamma} \text{Hull}(P \cdot x_0)$  is in  $\text{Hull}(P \cdot x_0)$  for  $P$  big enough, hence  $\varphi$  preserves the orientation. Finally, the actions  $\rho$  and  $\rho'$  being minimal, we have  $\bigcup_{P \subset \Gamma} \text{Hull}(P \cdot x_0) = T$  and  $\bigcup_{P \subset \Gamma} \text{Hull}(P \cdot x'_0) = T'$ ; whence  $\varphi: T \rightarrow T'$  is an orientation-preserving equivariant isometry, such that  $\varphi(x_0) = x'_0$ .  $\square$

Of course,  $f$  and  $g$  depend on  $x_0$ , anyway it follows from this proposition that the classes of actions of  $\Gamma$  on fat  $\mathbb{R}$ -trees not reduced to a point, up to orientation-preserving equivariant isometry, form a set, which we denote  $\mathcal{T}''(\Gamma)$ . We denote by  $\mathcal{T}^o(\Gamma)$  its subset formed by those  $(\rho, T) \in \mathcal{T}''(\Gamma)$  such that  $\min_{x_0 \in T} \max_{\gamma \in S} d_T(x_0, \gamma \cdot x_0) = 1$  and such that whenever  $\rho$  possesses at least one global fixed point in  $\partial_\infty T$ , the tree  $T$  is isometric to  $\mathbb{R}$ .

Our aim now is to define a topology on the set  $m_\Gamma^o(2) \cup \mathcal{T}^o(\Gamma)$ .

### 3.2.2 Rigidity of the order

We first need to give some technical lemmas indicating that the orders given by triples of points in fat  $\mathbb{R}$ -trees, as well as in  $\mathbb{H}^2$ , are stable under small perturbations of the tree or of the plane. In all this section,  $X$  will be a fat  $\mathbb{R}$ -tree or the hyperbolic plane  $\mathbb{H}^2$ , equipped with its orientation (and hence, with a total cyclic order on its boundary), and with a metric  $\frac{d_{\mathbb{H}^2}}{d}$  proportional to its usual metric. Its best hyperbolic constant is then  $\delta(X) = \frac{\delta_{\mathbb{H}^2}}{d}$ .

**Lemma 3.11** *Let  $x_1, x_2, x_3 \in X$ . Then there exists a unique  $x_0 \in X$  which minimizes the function  $x \mapsto d(x, x_1) + d(x, x_2) + d(x, x_3)$ . Moreover, the function  $X^3 \rightarrow X$  defined by  $(x_1, x_2, x_3) \mapsto x_0$  is continuous.*

#### Proof

If  $X$  is an  $\mathbb{R}$ -tree, then we check easily that the unique point  $m \in X$  such that  $[x_1, x_2] \cap [x_1, x_3] = [x_1, m]$  is the point  $x_0$  wanted. If  $X$  is the hyperbolic plane  $\mathbb{H}^2$  equipped with a proportional metric to  $d_{\mathbb{H}^2}$ , then the function  $x \mapsto d(x, x_1) + d(x, x_2) + d(x, x_3)$  is convex, proper, hence achieves a minimum. And the  $CAT(0)$  inequality implies that this function cannot be constant on any non degenerate segment, hence this minimum is unique. Moreover, this convex function depends continuously on  $x_1, x_2$  and  $x_3$ , hence its unique minimum also depends continuously on  $x_1, x_2$  and  $x_3$ .  $\square$

**Remark 3.12** *In the Euclidean plane  $\mathbb{R}^2$ , the point  $x_0$  is called the Fermat point of the triangle  $\Delta(x_1, x_2, x_3)$ . If this triangle has angles smaller than  $\frac{2\pi}{3}$ , then this point coincides with the Torricelli point, which, in that case, sees every edge of the triangle under an angle equal to  $\frac{2\pi}{3}$  (see e.g. [Fre96]).*

Let  $A \geq 0$ . We denote by  $V(A) \subset X^3$  the set of  $(x_1, x_2, x_3) \in X^3$  such that for every permutation  $(i, j, k)$  of  $(1, 2, 3)$ , we have  $d(x_i, x_j) + d(x_j, x_k) - d(x_i, x_k) > 2A$ .

**Lemma 3.13** *There exists a universal constant  $C_0 \geq 0$  such that for all  $(x_1, x_2, x_3) \in V(C_0\delta(X))$ , we have  $x_0 \notin \{x_1, x_2, x_3\}$ , where  $x_0$  is as in the lemma 3.11.*

**Proof**

The proof in the case when  $X$  is an  $\mathbb{R}$ -tree follows from the same observation as in the proof of lemma 3.11. In that case, any  $C_0 \geq 0$  satisfies the lemma. Now, suppose that  $X = (\mathbb{H}^2, d_{\mathbb{H}^2})$ . First notice that if all the angles of a non degenerate triangle  $\Delta(x_1, x_2, x_3)$  are strictly lower than  $\frac{\pi}{2}$ , then  $x_0 \notin \{x_1, x_2, x_3\}$ . Indeed, suppose that the angle at  $x_1$  is smaller than  $\frac{\pi}{2}$ . Then, for some  $x'_3 \in [x_1, x_3]$ , such that  $x'_3 \neq x_1$ , the orthogonal projection  $p(x'_3)$  of  $x'_3$  on the geodesic  $(x_1, x_2)$  is strictly between  $x_1$  and  $x_2$ . Then  $d(x_1, p(x'_3)) + d(x_2, p(x'_3)) + d(x_3, p(x'_3)) < d(x_1, x_2) + d(x_1, x_3)$ , so that  $x_0 \neq x_1$ . Similarly  $x_0 \notin \{x_2, x_3\}$ . Now, we shall prove that the condition  $(x_1, x_2, x_3) \in V(A)$ , for  $A$  large enough, implies that all the angles of the triangle  $\Delta(x_1, x_2, x_3)$  are lower than  $\frac{\pi}{2}$ . Consider Poincaré's disk model of  $\mathbb{H}^2$ , and let  $a$  be the centre of the disk. Let  $u_1, u_2$  be two points of  $\partial\mathbb{H}^2$ , such that the measure of the angle  $\widehat{u_1 a u_2}$  is equal to  $\frac{\pi}{2}$ . Let  $A$  be the hyperbolic distance between  $a$  and the geodesic  $(u_1 u_2)$  (explicit computation yields  $A = \frac{1}{2} \ln 3$ , see e.g. [CDP89], lemme 4.1). Now suppose that in the triangle  $\Delta(x_1, x_2, x_3)$ , the angle at  $x_1$  is greater or equal to  $\frac{\pi}{2}$ . Then  $d(x_1, [x_2, x_3]) \leq A$ , and this implies that  $d(x_1, x_2) + d(x_1, x_3) - d(x_2, x_3) \leq 2A$ , so that  $(x_1, x_2, x_3) \notin V(A)$ . It follows that  $C_0 = \frac{A}{\delta_{\mathbb{H}^2}} = \frac{\ln 3}{2 \ln(1+\sqrt{2})}$  satisfies the lemma, in the case when  $X = (\mathbb{H}^2, d_{\mathbb{H}^2})$ . Finally, in the case when  $X$  is the hyperbolic plane equipped with a proportional metric to  $\frac{d_{\mathbb{H}^2}}{d}$ , the preceding arguments prove the lemma, with the same constant  $C_0$ .  $\square$

In the sequel,  $C_0$  will be the constant that has been explicited in the proof of lemma 3.13. By choosing another equivalent definition of the  $\delta$ -hyperbolicity in the sense of Gromov and by considering, for instance, the real number  $A = \frac{1}{2} \ln 3$  as the hyperbolicity constant of  $\mathbb{H}^2$ , we would have got  $C_0 = 1$ . Therefore, the value of  $C_0$  does not have any deep meaning here.

Now define a set  $U \subset X^6$ , as follows: we say that  $(x_1, x_2, x_3, y_1, y_2, y_3) \notin U$  if there exist  $i, j \in \{1, 2, 3\}$ ,  $i \neq j$ , and rays  $r_i, r_j$  issued from  $x_i, x_j$  and passing through  $y_i, y_j$  respectively, such that  $r_i, r_j$  represent the same point of  $\partial_\infty X$ , or end at the same end point of  $T$ , in the case when  $T$  is a tree (heuristically,  $(x_1, \dots, y_3) \in U$  if the oriented segments  $[x_i, y_i]$  point three distinct directions). This implies, in particular, that  $x_i \neq y_i$ , for all  $i$ . If  $X = \mathbb{H}^2$ , then there is a unique ray (up to parametrization) issued from  $x_i$

and passing through  $y_i$ ; the condition  $(x_1, \dots, y_3) \in U$  expresses the fact that the ends of these three rays are three distinct points of  $\partial\mathbb{H}^2$ . In the case of an  $\mathbb{R}$ -tree, we can also give the following equivalent condition:

**Lemma 3.14** *Let  $X$  be an  $\mathbb{R}$ -tree. Then  $(x_1, \dots, y_3) \in U$  if and only if for every  $i \in \{1, 2, 3\}$ , the points  $x_i$ ,  $y_{i+1}$  and  $y_{i+2}$  (we are using here a cyclic notation for the indices) are in the same connected component of  $X \setminus \{y_i\}$ .*

**Proof**

We first check that  $(x_1, \dots, y_3) \in U$  implies that  $y_1$ ,  $y_2$  and  $y_3$  are not aligned. Suppose that  $y_1$ ,  $y_2$  and  $y_3$  are three points pairwise distinct, and are aligned (the case when some of them coincide is treated similarly). For instance, take  $y_2 \in [y_1, y_3]$ . If  $x_1, x_3$  are in the same connected component of  $X \setminus \{y_1, y_3\}$  as  $y_2$ , and if  $x_2$  is not in the same connected component as  $y_1$  then there exist rays  $r_1, r_2$  issued from  $x_1, x_2$  and passing through  $y_1, y_2$  respectively, such that  $r_2$  passes through  $y_1$ , and such that  $r_1, r_2$  are identical after their passage at  $y_1$ . The point  $x_2$  cannot be in several connected components at the same time so in that case we have  $(x_1, \dots, y_3) \notin U$ . Suppose then that  $x_1$  and  $y_2$  are in two distinct components of  $X \setminus \{y_1\}$ . Then in order to have  $(x_1, \dots, y_3) \in U$ , the point  $x_3$  cannot be in the same connected component of  $X \setminus \{y_3\}$  as  $y_1$ . As in the preceding case, we cannot place the point  $x_2$ , once again, so that  $(x_1, \dots, y_3)$  is in  $U$ .

In order to have  $(x_1, \dots, y_3) \in U$  it is necessary that  $y_1, y_2, y_3$  form a non degenerate tripod. If, for instance,  $x_1$  and  $y_2, y_3$  are in two distinct connected components of  $X \setminus \{y_1\}$ , then: either  $x_2$  is in the same component as  $X \setminus \{y_2\}$  as  $y_1, y_3$  and in that case we construct rays  $r_1, r_2$  both passing through  $y_2$  and identical after their passage at  $y_2$ ; either  $x_2$  is in another connected component of  $X \setminus \{y_2\}$  as  $y_1, y_3$  and then we construct rays  $r_1, r_2$  which are identical after their passage at  $y_3$ . The only remaining case is when for every  $i$ ,  $x_i$  is in the same connected component of  $X \setminus \{y_i\}$  as  $y_1, y_2$  and in that case, regardless to the three rays  $r_i$  issued from  $x_i$  and passing through  $y_i$ , the three rays  $r_1, r_2$  and  $r_3$  leave the tripod at the respective vertices  $y_1, y_2, y_3$  and cannot go towards a same end (or end point) of  $\partial_\infty X$ .  $\square$

**Lemma 3.15** *If  $(x_1, \dots, y_3) \in U$  and if  $r_i, r'_i$  are rays issued from  $x_i$  and passing through  $y_i$ , then  $o(r_1, r_2, r_3) = o(r'_1, r'_2, r'_3)$ .*

We write  $o(x_1, \dots, y_3) = o(r_1, r_2, r_3)$  in that case.

**Proof**

We have  $x_i \neq y_i$  so in the case when  $X = \mathbb{H}^2$ , there exists a unique ray  $r_i$  issued from  $x_i$  and passing through  $y_i$ . Now suppose that  $X$  is a fat  $\mathbb{R}$ -tree. By lemma 3.14,  $y_1, y_2$  and  $y_3$  are not aligned; let then  $y_0$  be such that  $[y_1, y_2] \cap [y_1, y_3] = [y_1, y_0]$ . Then  $y_1, y_2, y_3$  define three distinct germs of rays issued from  $y_0$ . Still by lemma 3.14, the condition  $(x_1, \dots, y_3) \in U$  implies that for every  $i$ ,  $x_i$  lies in the connected component of  $X \setminus \{y_i\}$  containing  $y_0$ . In particular, every ray  $r$  issued from  $x_i$  and passing through  $y_i$

defines a unique ray issued from  $y_0$  and passing through  $y_i$ : it is the ray joining  $y_0$  to  $y_i$ , and which then continues as the ray  $r$ . Therefore, the equality  $o(r_1, r_2, r_3) = o(r'_1, r'_2, r'_3)$  follows from the coherence condition on the order  $o$  of the fat  $\mathbb{R}$ -tree we are considering.  $\square$

**Lemma 3.16** *The function  $o: U \rightarrow \{-1, 1\}$  thereby defined is continuous.*

**Proof**

If  $X = \mathbb{H}^2$ , it is immediate that the function  $X^2 \setminus \Delta \rightarrow \partial X$  (where  $\Delta$  is the diagonal) sending  $(x, y)$  on the end of the ray issued from  $x$  and passing through  $y$  is continuous (where  $\partial X = \mathbb{S}^1$  is equipped with the usual topology), and hence  $o: U \rightarrow \{-1, 1\}$  is simply the composition of two continuous functions. In the case when  $X$  is a fat  $\mathbb{R}$ -tree, the proof is similar to that of the preceding lemma.  $\square$

**Lemma 3.17** *Let  $x_1, \dots, y_3$  such that for every  $i \in \{1, 2, 3\}$ , we have  $\sum_j d(x_i, y_j) < \sum_j d(y_i, y_j)$ . Then  $(x_1, \dots, y_3) \in U$ .*

**Proof**

First suppose that  $X$  is an  $\mathbb{R}$ -tree. By lemma 3.14, if  $(x_1, \dots, y_3) \notin U$ , then for some  $i$ , either  $x_i$  is not in the same component of  $X \setminus \{y_i\}$  as  $y_{i+1}$  and  $y_{i+2}$ , and in that case  $\sum_j d(x_i, y_j) = \sum_j d(y_i, y_j) + 3d(x_i, y_i) > \sum_j d(y_i, y_j)$ , either  $y_i$  lies in the segment  $[y_{i+1}, y_{i+2}]$  and then  $y_i$  realizes the minimum, in  $X$ , of the function  $x \mapsto d(x, y_1) + d(x, y_2) + d(x, y_3)$ , so that  $\sum_j d(x_i, y_j) \geq \sum_j d(y_i, y_j)$ .

Now consider the case when  $X$  is the hyperbolic plane, and suppose that  $x_i \neq y_i$  for every  $i$ , and that  $(x_1, \dots, y_3) \notin U$ . Suppose, for instance, that the rays  $[x_1, y_1]$  and  $[x_2, y_2]$  have the same end. Suppose also, by contradiction, that  $\sum_j d(x_i, y_j) < \sum_j d(y_i, y_j)$ . Then  $d(x_1, y_1) + d(x_1, y_2) + d(x_1, y_3) < d(y_1, y_2) + d(y_1, y_3)$  hence  $d(x_1, y_2) < d(y_1, y_2)$ , and, similarly,  $d(x_2, y_1) < d(y_1, y_2)$ . Since these inequalities are strict, we can move slightly the point  $x_2$  so that the rays  $[x_1, y_1]$  and  $[x_2, y_2]$  intersect at some point  $a \in \mathbb{H}^2$ , without losing these inequalities. By putting  $a$  at the centre of Poincaré's disk model of  $\mathbb{H}^2$ , we see that the condition  $d(x_1, y_2) < d(y_1, y_2)$  implies  $d(a, y_1) < d(a, y_2)$ . Indeed,  $y_2$  is closer to  $x_1$  than to  $y_1$ , hence  $y_2$  lies in the hyperbolic half-plane determined by the mediator of the segment  $[x_1, y_1]$  which does not contain  $a$ , and this half-plane does not meet the ball  $B(a, d(a, y_1))$ . Similarly, we have  $d(a, y_2) < d(a, y_1)$ , hence a contradiction.  $\square$

In particular, if  $(x_1, x_2, x_3) \in V(C_0\delta(X))$  and if  $x_0$  realizes the minimum of the function  $x \mapsto d(x, x_1) + d(x, x_2) + d(x, x_3)$ , then we have  $(x_0, x_0, x_0, x_1, x_2, x_3) \in U$ . We set  $o(x_1, x_2, x_3) = o(x_0, \dots, x_3)$  in that case.

**Remark 3.18** *Let  $y_1, y_2, y_3 \in X$ . The hypotheses of lemma 3.17 being convex conditions on  $x_1, x_2$  and  $x_3$ , they define convex subsets of  $X$ . In particular, by lemma 3.16, if  $(y_1, y_2, y_3) \in V(C_0\delta(X))$  and if for every  $i \in \{1, 2, 3\}$ ,  $\sum_j d(x_i, y_j) < \sum_j d(y_i, y_j)$ , then we have  $o(x_1, \dots, y_3) = o(y_1, y_2, y_3)$ .*

**Remark 3.19** Suppose that the spaces  $X$  and  $X'$  are the hyperbolic plane (equipped with a proportional distance to the usual distance) or an  $\mathbb{R}$ -tree, and let  $x_1, x_2, x_3 \in X$ ,  $x'_1, x'_2, x'_3 \in X'$ . Suppose that  $(x_1, x_2, x_3) \in V(C_0(\delta(X) + a) + b)$ ,  $|\delta(X) - \delta(X')| < \varepsilon_1$ , and  $|d(x_i, x_j) - d(x'_i, x'_j)| < 2\varepsilon_2$  for all  $i, j \in \{1, 2, 3\}$ . Then we have  $(x'_1, x'_2, x'_3) \in V(C_0(\delta(X') + a - \varepsilon_1) + b - 3\varepsilon_2)$ .

Let  $(\rho, X, o), (\rho', X', o') \in m_\Gamma^o(2) \cup \mathcal{T}^o(\Gamma)$ , and let  $\varepsilon > 0$ ,  $K = \{x_1, \dots, x_p\} \subset X$ , and let  $P$  be any finite subset of  $\Gamma$ . Suppose that  $(\rho', X') \in U''_{K, \varepsilon, P}(\rho, X)$ . This means (see section 2.2.1) that there exists a collection  $K' = \{x'_1, \dots, x'_p\} \subset X'$  such that for every  $g, h \in P$ , and every  $i, j \in \{1, \dots, p\}$  we have

$$|d(\rho(g) \cdot x_i, \rho(h) \cdot x_j) - d'(\rho'(g) \cdot x'_i, \rho'(h) \cdot x'_j)| < \varepsilon \text{ et } |\delta(X) - \delta(X')| < \varepsilon.$$

**Definition 3.20** If  $o(x_i, x_j, x_k) = o(x'_i, x'_j, x'_k)$  for every  $i, j, k \in \{1, \dots, p\}$  such that  $(x_i, x_j, x_k) \in V(C_0(\delta(X) + \varepsilon) + 3\varepsilon)$ , we say that  $K$  and  $K'$  come in the same order.

Note that if  $(x_i, x_j, x_k) \in V(C_0(\delta(X) + \varepsilon) + 3\varepsilon)$ , then  $(x'_i, x'_j, x'_k) \in V(C_0\delta(X'))$  and hence  $o(x'_i, x'_j, x'_k)$  is well-defined, so that definition 3.20 makes sense.

### 3.2.3 Oriented equivariant Gromov topology

Let  $(\rho, X) \in m_\Gamma^o(2) \cup \mathcal{T}^o(\Gamma)$ , let  $\varepsilon > 0$ ,  $K = \{x_1, \dots, x_p\} \subset X$  and let  $P$  be a finite subset of  $\Gamma$ . We denote by  $U''_{K, \varepsilon, P}(\rho, X)$  the set consisting of the  $(\rho', X') \in m_\Gamma^o(2) \cup \mathcal{T}^o(\Gamma)$  such that there exists a collection  $K' = \{x'_1, \dots, x'_p\} \subset X'$  such that for every  $g, h \in P$ , and every  $i, j \in \{1, \dots, p\}$  we have

$$|d(\rho(g) \cdot x_i, \rho(h) \cdot x_j) - d'(\rho'(g) \cdot x'_i, \rho'(h) \cdot x'_j)| < \varepsilon \text{ et } |\delta(X) - \delta(X')| < \varepsilon,$$

and such that  $K$  and  $K'$  come in the same order.

**Proposition 3.21** The sets  $U''_{K, \varepsilon, P}(\rho, X)$  form the basis of open sets of some topology; we call it the oriented equivariant Gromov topology.

#### Proof

Of course, we always have  $(\rho, X) \in U''_{K, \varepsilon, P}(\rho, X)$ . Hence, we need only check that if  $(\rho, X) \in U''_{K_1, \varepsilon_1, P_1}(\rho_1, X_1) \cap U''_{K_2, \varepsilon_2, P_2}(\rho_2, X_2)$  then, for some  $K$ ,  $\mu > 0$  and  $P$  we have  $U''_{K, \mu, P}(\rho, X) \subset U''_{K_1, \varepsilon_1, P_1}(\rho_1, X_1) \cap U''_{K_2, \varepsilon_2, P_2}(\rho_2, X_2)$ . It is just a technical verification. Denote  $K_1 = \{a'_1, \dots, a'_{n_1}\}$  and  $K_2 = \{b'_1, \dots, b'_{n_2}\}$ . Then there exists a finite collection  $K = \{a_1, \dots, a_{n_1}, b_1, \dots, b_{n_2}\} \subset X$  such that:

$$\forall i, j \leq n_1, \forall \gamma_1, \gamma_2 \in P_1, |d_X(\rho(\gamma_1) \cdot a_i, \rho(\gamma_2) \cdot a_j) - d_{X_1}(\rho_1(\gamma_1) \cdot a'_i, \rho_1(\gamma_2) \cdot a'_j)| < \varepsilon_1,$$

$|\delta(X) - \delta(X_1)| < \varepsilon_1$ , and such that  $o_{X_1}(a'_i, a'_j, a'_k) = o_X(a_i, a_j, a_k)$  for every  $i, j, k$  such that  $(a'_i, a'_j, a'_k) \in V(C_0(\delta(X_1) + \varepsilon_1) + 3\varepsilon_1)$ ; similarly  $o_{X_1}(b'_i, b'_j, b'_k) = o_X(b_i, b_j, b_k)$  for every  $i, j, k$  such that  $(b'_i, b'_j, b'_k) \in V(C_0(\delta(X_1) + \varepsilon_1) + 3\varepsilon_1)$ . Since all these *finitely many* inequalities are strict, they can all be refined simultaneously by some  $\mu > 0$ . In other words,

$$|d_X(\rho(\gamma_1) \cdot a_i, \rho(\gamma_2) \cdot a_j) - d_{X_1}(\rho_1(\gamma_1) \cdot a'_i, \rho_1(\gamma_2) \cdot a'_j)| < \varepsilon_1 - \mu, \quad (6)$$



$$|\delta(X) - \delta(X_1)| < \varepsilon_1 - \mu, \quad (7)$$

and the condition  $o_{X_1}(a'_i, a'_j, a'_k) = o_X(a_i, a_j, a_k)$  is verified for all

$$(a'_i, a'_j, a'_k) \in V(C_0(\delta(X_1) + \varepsilon_1 + \mu) + 3\varepsilon_1 + 3\mu). \quad (8)$$

Now, take  $(\rho'', X'') \in U''_{K, \mu, P_1 \cup P_2}(\rho, X)$ . Then in particular, for every  $i, j \leq n_1$  and every  $\gamma_1, \gamma_2 \in P_1$ ,

$$|d_{X''}(\rho''(\gamma_1) \cdot a''_i, \rho''(\gamma_2) \cdot a''_j) - d_X(\rho(\gamma_1) \cdot a_i, \rho(\gamma_2) \cdot a_j)| < \mu, \quad (9)$$

$|\delta(X) - \delta(X'')| < \mu$ , and for every  $i, j, k$  such that  $(a_i, a_j, a_k) \in V(C_0(\delta(X) + \mu) + 3\mu)$ ,  $o_X(a_i, a_j, a_k) = o_{X''}(a''_i, a''_j, a''_k)$ . Now, the conditions (6) and (9) imply that for every  $i, j \leq n_1$  and every  $\gamma_1, \gamma_2 \in P_1$ ,

$$|d_{X''}(\rho''(\gamma_1) \cdot a''_i, \rho''(\gamma_2) \cdot a''_j) - d_{X_1}(\rho_1(\gamma_1) \cdot a'_i, \rho_1(\gamma_2) \cdot a'_j)| < \varepsilon_1,$$

and, by remark 3.19, the conditions (6), (7) and (8) imply that

$$(a_i, a_j, a_k) \in V(C_0(\delta(X) + 2\mu) + \frac{3}{2}\varepsilon_1 + \frac{9}{2}\mu) \subset V(C_0(\delta(X) + \mu) + 3\mu),$$

so that  $o_{X''}(a''_i, a''_j, a''_k) = o_X(a_i, a_j, a_k) = o_{X_1}(a'_i, a'_j, a'_k)$ . In that way we get  $(\rho'', X'') \in U''_{K_1, \varepsilon_1, P_1}(\rho_1, X_1)$ , and, similarly,  $(\rho'', X'') \in U''_{K_2, \varepsilon_2, P_2}(\rho_2, X_2)$ .  $\square$

**Proposition 3.22** *The oriented equivariant Gromov topology coincides with the usual topology on  $m_\Gamma^o(2)$ .*

### Proof

It is the same proof as the one of F. Paulin's proposition 6.2 in [Pau88], with minor modifications. In that proof (recalled here, as proposition 2.16), the only difference is that the isometry  $\phi$  of 2.17, is now an orientation-preserving isometry.  $\square$

Of course, we denote by  $\overline{m_\Gamma^o(2)}$  the closure of  $m_\Gamma^o(2)$  in the space  $m_\Gamma^o(2) \cup T^o(\Gamma)$ , equipped with the oriented equivariant Gromov topology.

We also write  $\overline{m_\Gamma^u} = \overline{m_\Gamma^u(2)}$  and  $\overline{m_\Gamma^o} = \overline{m_\Gamma^o(2)}$ .

### 3.2.4 The space $\overline{m_\Gamma^o}$ is compact

Denote by  $\pi : \overline{m_\Gamma^o} \rightarrow \overline{m_\Gamma^u}$  the natural function consisting in forgetting the orientation.

**Proposition 3.23** *The map  $\pi$  is continuous, and its fibres are compact Hausdorff.*

### Proof

First, the continuity of  $\pi$  follows directly from the definition of these two topologies.

Now, let  $(\rho, T) \in \overline{m_\Gamma^u}$ . If  $T$  is  $\mathbb{H}^2$ , then the fibre  $\pi^{-1}(\rho, T)$  has cardinal 2 in the Hausdorff space  $m_\Gamma^o$  (by theorem 2.5), hence it is compact Hausdorff. Suppose now that  $T$  is an  $\mathbb{R}$ -tree. By definition, the set  $\pi^{-1}(\rho, T)$  is a subset of  $O_T$ . By definition of the sets  $U''_{K,\varepsilon,P}(\rho, T)$ , the induced topology on  $\pi^{-1}(\rho, T)$  in  $O_T$  coincides with the oriented equivariant Gromov topology (in each of these topologies, open sets are defined by equalities of  $o$  on finite subsets of  $Trip(T)$ ).

We shall notice now that  $\overline{m_\Gamma^o}$  is Hausdorff. Indeed, if  $(\rho, X, o)$  and  $(\rho', X', o')$  are distinct and are not separated by open sets, then  $\delta(X) = \delta(X')$ . The open set  $m_\Gamma^o$  being Hausdorff, this means that  $X$  and  $X'$  are fat  $\mathbb{R}$ -trees. Since the map  $\pi: \overline{m_\Gamma^o} \rightarrow \overline{m_\Gamma^u}$  is continuous, this implies that these two spaces differ only by the orientation: but by definition of the oriented equivariant Gromov topology, there are two open sets separating  $(\rho, X, o)$  and  $(\rho', X', o')$ .

Consequently, we need only prove that  $\pi^{-1}(\rho, T)$  is closed in  $O_T$ . Let  $E_T''$  be the set of orders  $o: (\partial T)^3 \rightarrow \{-1, 0, 1\}$  which satisfy the hypotheses of definition 2.20, and which are invariant by  $\rho$ . These are closed conditions, hence  $E_T''$  is closed in  $f(O_T)$ . By definition,  $E_T'' \subset \mathcal{T}^o(\Gamma)$ , and  $\overline{m_\Gamma^o} \cap \mathcal{T}^o(\Gamma)$  is closed in  $\mathcal{T}^o(\Gamma)$ , hence  $\pi^{-1}(\rho, T)$  is closed in  $E_T''$ . It follows that  $\pi^{-1}(\rho, T)$  is closed in a compact Hausdorff space, hence it is compact.  $\square$

**Theorem 3.24** *The space  $\overline{m_\Gamma^o}$ , equipped with the function  $m_\Gamma^o \hookrightarrow \overline{m_\Gamma^o}$ , is a natural compactification of  $m_\Gamma^o$ .*

Here again, after [Pau04], by “natural”, we mean that the action of  $Out(\Gamma)$  on  $m_\Gamma^o$  extends continuously to an action of  $Out(\Gamma)$  on  $\overline{m_\Gamma^o}$ .

**Proof**

Since  $m_\Gamma^o$  is open and dense in  $\overline{m_\Gamma^o}$  and since, by definition of the oriented equivariant Gromov topology, the action of  $Out(\Gamma)$  on  $\overline{m_\Gamma^o}$  is continuous, it suffices to prove that the space  $\overline{m_\Gamma^o}$  is compact Hausdorff. We have already seen that  $\overline{m_\Gamma^o}$  is Hausdorff, and by the definition of compactity in terms of ultrafilters (see e.g. [Bou71], page 59), we need only prove that every ultrafilter in  $\overline{m_\Gamma^o}$  converges.

Let  $\omega$  be an ultrafilter in  $\overline{m_\Gamma^o}$ . Then the image of the ultrafilter  $\pi(\omega)$  is an ultrafilter in the compact space  $\overline{m_\Gamma^u}$  (see e.g. [Bou71], proposition 10, page 41) hence it converges to some action  $(\rho_\infty, X_\infty) \in \overline{m_\Gamma^u}$ . If  $X_\infty = \mathbb{H}^2$ , then it follows from proposition 3.22 that  $X$  is equipped with an orientation, compatible with the convergence of the ultrafilter. We need only prove that if  $X_\infty$  is an  $\mathbb{R}$ -tree (denote  $(\rho, T) = (\rho_\infty, X_\infty)$  in that case) then there exists a coherent order  $o: (\partial_\infty T)^3 \rightarrow \{-1, 0, 1\}$  which verifies the hypotheses of definition 2.20, which is invariant under the action of  $\Gamma$ , and such that  $(\rho, T)$ , equipped with this order, is indeed the limit, in  $\overline{m_\Gamma^o}$ , of the ultrafilter  $\omega$ .

Consider an increasing sequence  $T_k \subset T$  of finite, closed subtrees of  $T$ , such that  $\bigcup_k T_k = T$ . Suppose for simplicity that  $T_1$  is a singleton  $\{x_0\}$ . For all  $k \geq 1$ ,  $N \geq 0$ , denote by  $F_{k,N}$  the finite collection of elements of  $T_k$  consisting of all the end points of  $T_k$ , as well as all the points of  $T_k$  whose distance to

$x_0$  is a multiple of  $\frac{1}{2^N}$ . Since the ultrafilter  $\pi(\omega)$  converges to  $(\rho, T)$ , for all  $k, N$  and  $\varepsilon > 0$ , and for every finite subset  $P$  of  $\Gamma$ , for all  $M \in \omega$ , there exists  $(\rho_M, X_M, o_M) \in M$  such that  $(\rho_M, X_M) \in U'_{F_{k,N,\varepsilon,P}}(\rho, T)$ , i.e. there exists a  $P$ -equivariant  $\varepsilon$ -approximation between  $F_{k,N}$  and some finite collection  $K_M$  in  $X_M$ , for some  $(\rho_M, X_M, o_M) \in M$ . For all  $M \in \omega$ , denote by  $O_{k,N,\varepsilon,P,M}$  the set of functions  $o \in O_T$  such that there exists such an approximation, and such that for every  $x_1, x_2, x_3 \in F_{k,N}$  with  $(x_1, x_2, x_3) \in V\left((C_0 + 3\varepsilon) + \frac{1}{2^N}\right)$ , and for all corresponding  $x'_1, x'_2, x'_3 \in K_M$ , we have  $o(\text{Trip}(x_1, x_2, x_3)) = o_M(x'_1, x'_2, x'_3)$  (note that this is indeed well defined, thanks to remark 3.18).

We cut the end of the proof into the two following lemmas:

**Lemma 3.25** *For every  $k, N, \varepsilon$ ,  $P$  finite subset of  $\Gamma$  and every  $M \in \omega$ , the set  $O_{k,N,\varepsilon,P,M}$  is closed, and non empty. Moreover, if  $k > k'$ ,  $N > N'$ ,  $\varepsilon < \varepsilon'$ ,  $P' \subset P$  and  $M \subset M'$ , then  $O_{k,N,\varepsilon,P,M} \subset O_{k',N',\varepsilon',P',M'}$ .*

By compacity of  $E'_T$ , it follows that  $\bigcap_{k,N,\varepsilon,P,M} O_{k,N,\varepsilon,P,M} \neq \emptyset$ .

**Lemma 3.26** *Let  $o \in \bigcap_{k,N,\varepsilon,P,M} O_{k,N,\varepsilon,P,M}$ . Then  $o$  satisfies the conditions of definition 2.20. Hence,  $o$  defines a coherent total cyclic order. Moreover, it is invariant under the action of  $\Gamma$ , and the element  $(\rho, T)$ , equipped with the orientation  $o$ , is the limit, in  $\overline{m}_T^o$ , of the ultrafilter  $\omega$ .*

□

**Proof** of lemma 3.25.

It follows from the definition that  $O_{k,N,\varepsilon,P,M} \subset O_{k',N',\varepsilon',P',M'}$  if  $k > k'$ ,  $\varepsilon < \varepsilon'$ ,  $M \subset M'$  and  $P' \subset P$ .

The hypotheses concern only  $\text{Trip}(T_k)$ , which is a *finite* subset of  $\text{Trip}(T)$ , and hence  $O_{k,N,\varepsilon,P,M}$  is closed. We need to prove that it is also non empty. Up to restricting ourselves to a subset of it, we may suppose that  $\varepsilon$  is small enough and that  $N$  is large enough so that for every triple  $u_1, u_2, u_3$  of non aligned branch points of  $T_k$ , we have  $(u_1, u_2, u_3) \in V\left((C_0 + 3)\varepsilon + \frac{1}{2^N}\right)$ . Choose a (non-oriented)  $\varepsilon$ -approximation between  $E_{k,N}$  and  $K_M \subset X_M$  (such an approximation exists, since  $\pi(\omega)$  converges to  $(\rho, T)$  in  $\overline{m}_g^u$ ). For each tripod  $\text{Trip}(a_0, a_1, a_2, a_3) \in \text{Trip}(T_k)$  (where  $a_0$  is the center of the tripod), the only obstruction to defining  $o(\text{Trip}(a_0, a_1, a_2, a_3)) \in \{-1, 1\}$  according to  $K_M$ , would be that there exist two representants  $(a_0, a_1, a_2, a_3)$  and  $(a_0, b_1, b_2, b_3)$  of this same class of tripods, such that the triples  $(a_1, a_2, a_3)$  and  $(b'_1, b'_2, b'_3)$  are in  $V\left((C_0 + 3)\varepsilon + \frac{1}{2^N}\right)$ , and such that  $o_{X_M}(a'_1, a'_2, a'_3) = 1 = -o_{X_M}(b'_1, b'_2, b'_3)$ . Hence we need to prove that this is impossible. The condition  $(u_1, u_2, u_3) \in V\left((C_0 + 3)\varepsilon + \frac{1}{2^N}\right)$  for all the non aligned branched points of  $T_k$ , implies

that it is possible to go from the triple  $(a_1, a_2, a_3)$  to the triple  $(b_1, b_2, b_3)$  by a sequence of moves consisting of replacing  $a_1$ ,  $a_2$  or  $a_3$  by one of its close neighbours in  $T_k$ , without leaving  $V\left((C_0 + 3)\varepsilon + \frac{1}{2^N}\right)$ . In particular, we may suppose that  $(b_1, b_2, b_3) = (b_1, a_2, a_3)$  and  $d_T(a_1, b_1) \leq \frac{1}{2^N}$ . This implies that  $d_{X_M}(a'_1, b'_1) < \varepsilon + \frac{1}{2^N}$ . Take  $a'(t) \in [a'_1, b'_1]$ , with  $a'(0) = a'_1$  and  $a'(1) = b'_1$ . Then, for all  $t \in [0, 1]$ , and every permutation  $(i, j, k)$  of  $(1, 2, 3)$ , we have  $d(a'_i, a'_j) + d(a'_j, a'_k) - d(a'_i, a'_k) > 2\left(\left((C_0 + 3)\varepsilon + \frac{1}{2^N}\right) - 3\varepsilon - \frac{1}{2^N}\right)$ , so that for all  $t \in [0, 1]$ , we have  $(a'(t), a'_2, a'_3) \in V(C_0\delta(X))$ . It follows from the continuity of the order (and more precisely, from lemmas 3.11 and 3.16) that  $o_{X_M}(a'_1, a'_2, a'_3) = o_{X_M}(b'_1, a'_2, a'_3)$ .  $\square$

**Proof** of lemma 3.26.

It follows from the definitions of the sets  $O_{k,N,\varepsilon,P,M}$  that the order  $o : (\partial_\infty T)^3 \rightarrow \{-1, 1\}$  indeed defines an orientation, i.e., it is coherent, and satisfies the first two conditions of definition 2.20. The third condition, as well as the invariance of  $o$  under the action of  $\Gamma$ , are proved by considering a big enough subtree  $T_k$  of  $T$  containing the desired branched points, and by deriving the properties of  $o$  from the corresponding properties for  $o_M$ , which are supposed to be true since  $(\rho_M, X_M, o_M) \in \overline{m_\Gamma^o}$ . As an example we prove that  $o$  satisfies the third condition of definition 2.20; the proof of the invariance of  $o$  under  $\Gamma$  is similar. Let  $x_1, x_2, x_3, x_4 \in \partial_\infty T$  be such that  $o(x_1, x_2, x_3) = o(x_1, x_3, x_4) = 1$ ; we want to prove that  $o(x_1, x_2, x_4) = 1$ . Let  $k$  and  $N$  be sufficiently large so that  $T_k$  contains five points  $a_0, a_1, a_2, a_3, a_4 \in F_{k,N}$  such that every triple  $i, j, k$  of distinct elements of  $\{1, 2, 3, 4\}$ , we have  $(a_0, a_0, a_0, a_i, a_j, a_k) \in U$  in  $T_k$ , and such that for all  $i \in \{1, 2, 3, 4\}$ , the ray issued from  $a_0$  and passing through  $x_i$  also passes through  $a_i$ , and such that  $a_0 \in \text{Conv}(a_1, a_2, a_3, a_4)$  (such points indeed exist in  $T$ ). Then, for all  $\varepsilon$ -approximation between  $F_{k,N}$  and  $K_M \subset X_M$ , with  $\varepsilon$  small enough, we have  $(a'_0, a'_0, a'_0, a'_i, a'_j, a'_k) \in U$  in  $X_M$ , and then the equality  $o(x_1, x_2, x_4) = 1$  indeed follows from the fact that  $o_n$  satisfies the third condition of definition 2.20.

Now, for all  $M \in \omega$ , we have  $U''_{F_{k,N,\varepsilon,P}}(\rho, T, o) \cap M \neq \emptyset$  (for all  $\varepsilon'$  such that  $(C_0 + 3)\varepsilon < (C_0 + 3)\varepsilon' + \frac{1}{N}$ ). This means, by definition, that the point  $(\rho, T, o) \in \overline{m_\Gamma^o}$  is adherent to the filter  $\omega$ , and since  $\omega$  is an ultrafilter this implies that  $\omega$  converges to  $(\rho, T, o)$ .  $\square$

**Corollary 3.27** *The map  $\pi : \overline{m_\Gamma^o} \rightarrow \overline{m_\Gamma^u}$  is onto.*

**Proof**

Of course,  $m_\Gamma^o \rightarrow m_\Gamma^u$  is onto. Now, let  $T \in \partial \overline{m_\Gamma^u}$ , and  $\rho_n \in m_\Gamma^o$  be such that  $\pi(\rho_n)$  converges to  $T$ . Then  $\rho$  possesses a subsequence converging to some  $\rho_\infty$ , and by continuity of  $\pi$  we have  $\pi(\rho_\infty) = T$ .  $\square$

**Remark 3.28** *It is possible to write this proof without using ultrafilters (see [Wol07]), and to prove the sequential compactness first (by considering a sequence instead of an ultrafilter on  $\overline{m_g^o}$ ), and then to prove the compactness of  $\overline{m_g^o}$  by using elementary general topology.*

### 3.2.5 The space $\overline{m_g^o}$ possesses $4g - 3$ connected components

We are now going to focus on the case when  $\Gamma = \pi_1 \Sigma_g$ . We denote  $\mathcal{T}_g = \mathcal{T}(\pi_1 \Sigma_g)$ , and  $\mathcal{T}_g^o = \mathcal{T}(\pi_1 \Sigma_g)^o$ . If  $(\rho, T) \in \mathcal{T}_g^o$ , the set  $\partial_\infty T$  is equipped with a total cyclic order, preserved by the action of  $\pi_1 \Sigma$ , and hence it possesses an Euler class, as defined in section 2.3. Notice that if  $T$  is a line, then it follows from the definition of the Euler class that  $e(\rho, T) = 0$ .

**Theorem 3.29** *The Euler class  $e : m_g^o \cup \mathcal{T}_g^o \rightarrow \mathbb{Z}$  is a continuous function.*

This proof will use the technical statements established in section 2.3, which imply that we need only finitely many information about the order in order to compute the Euler class of a representation. We will be using here the notations introduced in that section.

#### Proof

First, the set  $m_g^o = \{(\rho, X) | \delta(X) \neq 0\}$  is open in  $m_g^o \cup \mathcal{T}_g^o$ , and it follows from the formula (1), in section 2.3.4, that  $e$  is continuous on  $m_g^o$ .

Now take an element  $(\rho_T, T) \in \overline{m_g^o} \setminus m_g^o$ . We shall prove that there exists a neighbourhood of  $T$ , in the sense of the oriented equivariant Gromov topology, consisting only in representations of the same Euler class as  $T$ . First suppose that  $T$  is not reduced to a line, so that it has at least three ends (in that case,  $T$  possesses infinitely many ends). Take  $x, y \in \partial_\infty T$ , such that  $x \notin P_{ref} \cdot y$ . We may suppose that  $\text{Card}(P_{ref} \cdot y) \geq 2$ ; otherwise  $\pi_1 \Sigma_g$  would fix every end of  $T$ , and, by minimality,  $T$  would be a point or a line.

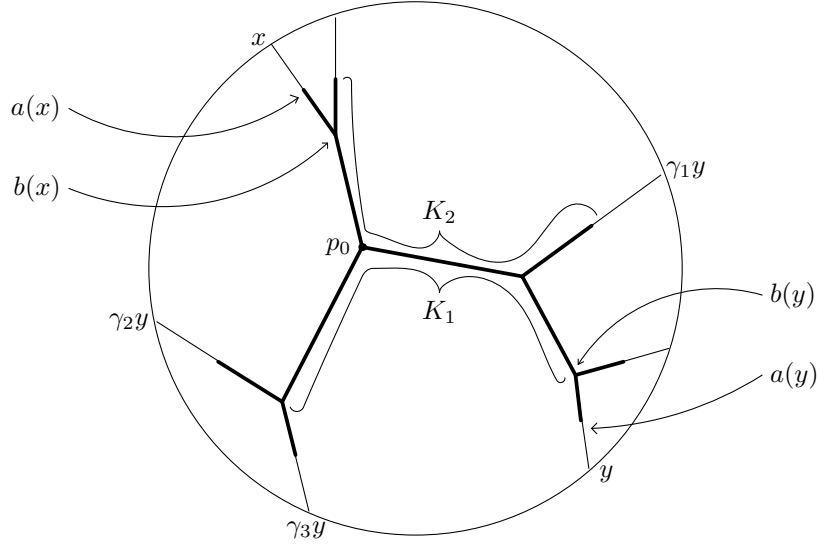
Every triple  $\{a, b, c\}$  of pairwise distinct elements of  $P_{ref} \cdot \{x, y\}$  determines a unique class of tripods in  $T$ ; we will denote by  $P_{abc}$  the centre of this tripod. Let  $K_1$  be the convex hull

$$K_1 = \text{Hull} \left\{ P_{abc} \mid (a, b, c) \in (P_{ref} \cdot \{x, y\})^3, \text{Card}\{a, b, c\} = 3 \right\}.$$

Put  $d_T = \max\{d(p, \gamma p) \mid p \in K_1, \gamma \in P_{ref}\}$  and let  $d_{K_1}$  be the diameter of  $K_1$ . Also, for every non degenerate triple  $\{a, b, c\} \subset P_{ref} \cdot \{x, y\}$ , let  $P_{abc}^a \subset (a, b) \cap (a, c)$  be such that  $d(P_{abc}^a, P_{abc}) = L$ , with  $L > 9d_T + 3d_{K_1}$ , and let  $K_2$  be the convex hull

$$K_2 = \text{Hull} \left\{ P_{abc}^a \mid (a, b, c) \in (P_{ref} \cdot \{x, y\})^3, \text{Card}\{a, b, c\} = 3 \right\}.$$

Finally, fix a point  $p_0 \in K_1$ .



Let  $a(x) = \bigcap_{p \in K_2} [p, x)$ . This is the “closest point to  $x$ ” in  $K_2$ . And let  $b(x)$  be the projection of  $a(x)$  on  $K_1$ . We define  $a(y)$  and  $b(y)$  similarly. We put  $K = \{p_0, a(x), b(x), a(y), b(y)\}$  and we consider  $(\rho, X) \in U''_{K, \frac{d_T}{C_0+3}, P_{ref}}(\rho_T, T)$ . We shall prove that  $(\rho_T, T)$  and  $(\rho, X)$  have the same Euler class, by applying proposition 2.36. In the space  $X$ , denote by  $p'_0, a'(y), b'(y), a'(x), b'(x)$  the corresponding points. Denote by  $x' \in \partial_\infty X$  the end of some ray  $[b'(x), a'(x))$  and by  $y' \in \partial_\infty X$  the end of some ray  $[b'(y), a'(y))$  (chosen arbitrarily, in the case of an  $\mathbb{R}$ -tree).

Let  $\gamma_1, \gamma_2, \gamma_3 \in P_{ref}$  be such that  $o(\gamma_1 x, \gamma_2 x, \gamma_3 y) = 1$ . We want to prove that  $o(\gamma_1 x', \gamma_2 x', \gamma_3 y') = 1$ . For this, we shall prove the three following equalities:

$$o(\gamma_1 x, \gamma_2 x, \gamma_3 y) = o(\gamma_1 a(x), \gamma_2 a(x), \gamma_3 a(y)), \quad (10)$$

$$o(\gamma_1 x', \gamma_2 x', \gamma_3 y') = o(\gamma_1 a'(x), \gamma_2 a'(x), \gamma_3 a'(y)), \quad (11)$$

$$o(\gamma_1 a(x), \gamma_2 a(x), \gamma_3 a(y)) = o(\gamma_1 a'(x), \gamma_2 a'(x), \gamma_3 a'(y)). \quad (12)$$

We can check that for every  $\gamma \in P_{ref}$ ,  $L - d_T \leq d(\gamma a(y), K_1) \leq L + d_T$ , and the centre of the tripod determined by  $\gamma_1 a(x)$ ,  $\gamma_2 a(x)$  and  $\gamma_3 a(y)$  lies in  $K_1$ , hence

$$(\gamma_1 a(x), \gamma_2 a(x), \gamma_3 a(y)) \in V(8d_T + 3d_{K_1}) \subset V(d_T),$$

so that all the terms of the equations (10), (11) and (12) are well-defined, and equation (12) holds. Put  $p_1 = \gamma_1 a(x)$ ,  $p'_1 = \gamma_1 a'(x)$ ,  $\dots$ ,  $p_3 = \gamma_3 a(y)$ ,  $p'_3 = \gamma_3 a'(y)$ . Then  $d_X(p'_0, p'_i) < d_{K_1} + L + d_T + \frac{d_T}{C_0+3}$ , and  $d_X(p'_i, p'_j) > 2L - 2d_T - \frac{d_T}{C_0+3}$ , so that for every  $i \in \{1, 2, 3\}$ ,  $\sum_j d_X(p'_0, p'_j) < \sum_j d_X(p'_i, p'_j)$ . These inequalities are even finer in  $T$ , and by lemma 3.17 and remark 3.18, the equalities (10) and (11) hold. Similarly, if  $\gamma_1, \gamma_2, \gamma_3 \in P_{ref}$  are such that  $o(\gamma_1 x, \gamma_2 y, \gamma_3 y) \neq 0$  then  $o(\gamma_1 x', \gamma_2 y', \gamma_3 y') = o(\gamma_1 x, \gamma_2 y, \gamma_3 y)$ , so that the

conditions of proposition 2.36 are satisfied. This finishes the proof, in the case when  $T$  is not a line.

Now, suppose that  $T$  is a line. We want to prove that there exists a neighbourhood of  $T$  consisting only in representations of Euler class zero. For simplicity we will prove the following:

**Lemma 3.30** *There exists a neighbourhood  $V'$  of  $T$  in which every fat  $\mathbb{R}$ -tree has Euler class zero.*

This lemma implies the theorem, for the following reason. For all  $k \neq 0$  such that  $|k| \leq 2g - 2$ , denote by  $L \subset \mathcal{T}_g^o$  the set of actions on lines, and denote by  $F_k \subset L$  the set of actions on lines  $(\rho, T)$  such that every neighbourhood of  $(\rho, T)$  contains representations of Euler class  $k$ . By lemma 3.30, we have  $F_k = L \cap \overline{m_{g,k}^o}$ . Hence,  $F_k$  is a closed subset of  $\partial m_{g,k}^o$  (indeed,  $L$  is a closed set, as, by definition of the topology, it is an open condition to contain a non degenerate tripod). Now, let  $(\rho, T) \in F_k$ . By lemma 3.30,  $(\rho, T)$  has a neighbourhood  $V \subset \mathcal{T}_g^o \cup m_g^o$  in which every fat  $\mathbb{R}$ -tree has Euler class 0. Put  $V' = V \cap \partial m_{g,k}^o$ . It is an open subset of  $\partial m_{g,k}^o$ . If there was a tree  $(\rho', T') \in V'$  not reduced to a line, then by the preceding argument,  $(\rho', T')$  has a neighbourhood consisting of actions (on hyperbolic planes or on trees) of Euler class 0, which is a contradiction since  $(\rho', T') \in \overline{m_{g,k}^o}$ . Hence,  $V'$  consists of actions on lines, i.e.,  $F_k$  is open in  $\partial m_{g,k}^o$ . By proposition 3.2, the space  $m_{g,k}^o$  is one-ended, hence  $\partial m_{g,k}^o$  is connected. And we can prove easily that  $\partial m_{g,k}^o$  contains actions not reduced to a line, hence  $F_k \neq \partial m_{g,k}^o$ . Whence,  $F_k = \emptyset$ , for all  $k \neq 0$ . In other words, every action on a line has a neighbourhood consisting of actions of Euler class 0; this finishes the proof of theorem 3.29.  $\square$

**Proof** of lemma 3.30.

Up to add some elements, we shall suppose here that the set  $P_{ref}$  contains  $S$  and that it is symmetric. Let  $(\rho, T)$  be a line, such that  $\min_{x_0 \in T} \max_{\gamma \in S} d(x_0, \gamma \cdot x_0) =$

1. Consider some point  $x_0 \in T$  as in the equality above. Let  $d_1$  be the greatest distance between  $x_0$  and  $\gamma x_0$ , for every  $\gamma \in P_{ref}$ . Consider points  $x_1, x_2, y_1, y_2$  of  $T$  such that  $x_i, y_i$  are on the same side of  $x_0$ , such that  $d(x_0, y_i) = d_1 + 4$  and  $d(x_0, x_i) = 2d_1 + 6$ . Let  $K$  be the finite subset of  $T$  consisting of  $x_1, x_2, y_1, y_2$  and  $P_{ref} \cdot x_0$ . Let  $(\rho', X') \in U''_{K, \frac{1}{6}, P_{ref}}(\rho, T)$ , we shall prove that  $e(\rho', T') = 0$ . If  $T'$  is a line, then there is nothing to do. Otherwise, for every point  $p'_i \in K'$  approximating  $K$ , denote by  $p''_i$  its projection on the segment  $[x'_1, x'_2]$ . This defines a new approximation, which realizes  $(\rho', T')$  as an element of  $U''_{K, 1, P_{ref}}(\rho, T)$ , and such that  $K''$  is contained in a segment. Let  $r$  be the end of a ray issued from  $x''_0$  and which leaves the segment  $[x'_1, x'_2]$  at a point  $p_0$  at distance at most 2 of  $x''_0$  (such a ray does exist, since  $\max_{\gamma \in S} d(x''_0, \gamma \cdot x''_0) < 2$ ). For every  $\gamma \in P_{ref}$ ,  $d(\gamma \cdot p_0, x''_0) < d_1 + 3$ , and hence the segment  $[x'_1, x'_2] \cap \text{Hull}(P_{ref} \cdot p_0)$  is contained (strictly, on each side), in the segment  $[y''_1, y''_2]$ . Similarly, for every  $\gamma \in P_{ref}$ ,  $[y'_1, y'_2] \subset [\gamma \cdot x'_1, \gamma \cdot x'_2]$ .

For  $i = 1, 2$ , let  $U_i$  be the set of ends of rays issued from  $x_0''$  and passing through  $x_i'$ , and let  $U_2'$  be the set of ends of rays issued from  $x_0''$  and passing through  $y_i''$ . Then  $P_{ref}$  sends  $U_1 \cup U_2$  on a subset of  $U_1' \cup U_2'$ . Then it follows from the coherence condition on the order on  $T'$  that for all  $x \in U_1'$ ,  $y \in U_2'$ ,  $o(r, x, y) \in \{-1, 1\}$  is constant. Suppose for instance that  $o(r, x, y) = 1$  for all  $x \in U_1'$ ,  $y \in U_2'$ . Thus, for every  $\gamma \in P_{ref}$ , the situation is the following.

- If  $\gamma$  sends  $U_1$  on a subset of  $U_1'$ , and  $U_2$  on a subset of  $U_2'$  (or, equivalently, if  $\rho(\gamma)$  preserves the orientation of  $T$ ), then  $o(\gamma \cdot r, x, y) = 1$  for all  $x \in U_1'$ ,  $y \in U_2'$ . Denote by  $A$  the set of these ends  $\gamma \cdot r$ .
- If  $\gamma$  sends  $U_1$  on a subset of  $U_2'$ , and  $U_2$  on a subset of  $U_1'$  (or, equivalently, if  $\rho(\gamma)$  reverses the orientation of  $T$ ), then  $o(\gamma \cdot r, x, y) = -1$  for all  $x \in U_1'$ ,  $y \in U_2'$ . Denote by  $B$  the set of such ends  $\gamma \cdot r$ .

Now equip the set  $\{a, u_1, b, u_2\}$  with the cyclic order in which we wrote them here. Denote by  $h$  the order-preserving bijection which exchanges  $a$  and  $b$  and exchanges  $u_1$  and  $u_2$ . Then we can consider the action  $\pi_1 \Sigma_g \rightarrow \{1, h\}$  on this ordered set, defined as follows: if  $\gamma \in \pi_1 \Sigma_g$  preserves the orientation of the line  $T$  then we send it to 1, otherwise we send it to  $h$ . Of course, this action has Euler class zero, and now it follows from proposition 2.36, which applies here, that  $e(\rho', T') = 0$ .  $\square$

### 3.3 Degeneracy of the non oriented compactification

#### 3.3.1 Non-orientable $\mathbb{R}$ -trees

Now we are going to prove that the existence of an orientation, on an  $\mathbb{R}$ -tree, preserved by the action of the group, is indeed a restrictive condition. More precisely:

**Proposition 3.31** *Let  $g \geq 3$ . Then the inclusion  $\overline{\partial m_g^u(2)} \subset \overline{\partial m_g^u(3)}$  is strict.*

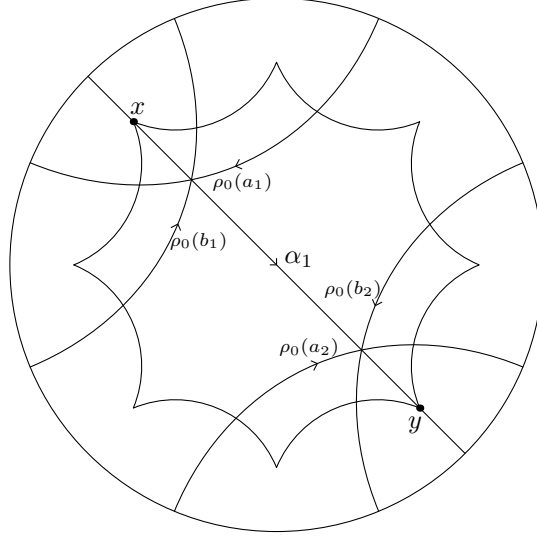
**Proof**

Of course, every isometric embedding of  $\mathbb{H}^2$  into  $\mathbb{H}^3$  gives rise to an embedding  $m_g^u(2) \subset m_g^u(3)$  and it follows that  $\overline{\partial m_g^u(2)} \subset \overline{\partial m_g^u(3)}$ . In order to prove that this inclusion is strict, we shall prove that there exists an element  $(T, \rho_\infty) \in \overline{\partial m_g^u(3)}$  such that no orientation on  $T$  is preserved by  $\rho_\infty$ . Since the map  $\pi: \overline{m_g^o(2)} \rightarrow \overline{m_g^u(2)}$  is onto, this implies that  $(T, \rho_\infty) \notin \overline{\partial m_g^u(2)}$ .

Consider the realization of  $\pi_1 \Sigma_2$  as a Fuchsian group acting on  $\mathbb{H}^2$  with a fundamental domain as symmetric as possible, that is, a regular octagon,



$$\rho_0: \pi_1 \Sigma_2 \rightarrow PSL(2, \mathbb{R}).$$



Denote by  $\rho_0(a_1), \dots, \rho_0(b_2) \in PSL(2, \mathbb{R})$  the hyperbolic isometries suggested in the above picture. The element  $\alpha_1 = \rho_0(a_1^{-1}b_1^{-1}a_2b_2)$  is hyperbolic, of axis  $(x, y)$  represented above (indeed,  $\alpha_1 \cdot x = y$ , and if  $\vec{u}$  is a unit tangent vector at  $x$  pointing towards  $y$ , the angles of  $\vec{u}$  and of its successive images with the edges of the octagon enable to check that the image of  $\vec{u}$  is again a vector whose direction is the one of the axis  $(x, y)$ , pointing in the opposite direction of  $x$ ). Note that  $\alpha_1$  is represented by a non separating simple closed curve on the surface  $\Sigma_2$  (in order to check that this curve is non separating, we can see for instance that the isometry  $\rho_0(a_1)$  identifies two points of the boundary of the fundamental domain represented by the above picture; these two points are in the two different sides of the axis of  $\alpha_1$ ; in other words: the axis of  $\rho_0(a_1)$  determines a simple closed curve on the surface, which intersects exactly once, transversely, the axis of  $\alpha_1$ , hence this axis defines a non separating curve). Thus, we can complete it into  $(\alpha_1, \beta_1, \alpha_2, \beta_2)$  represented by a system of curves on  $\Sigma_2$ , with  $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \rho_0(\pi_1 \Sigma_2)$ . Define then  $\rho_n: \pi_1 \Sigma_2 \rightarrow PSL(2, \mathbb{R})$  by the formulas  $\rho_n(a_i) = \alpha_i$ ,  $\rho_n(b_2) = \beta_2$ ,  $\rho_n(b_1) = \beta_1 \alpha_1^n$ , for all  $n \geq 1$ . Then  $\rho_n$  is faithful and discrete, and, as subgroups of  $PSL(2, \mathbb{R})$ , we have  $Im(\rho_n) = Im(\rho_0)$ . Since  $\rho_0$  is purely hyperbolic (i.e., every element of  $\pi_1 \Sigma_2 \setminus \{1\}$  is sent to a hyperbolic element), the elements  $\alpha_1, \beta_1 \in Hyp$  do not share any fixed points on  $\partial \mathbb{H}^2$ . Hence  $Tr(\beta_1 \alpha_1^n) \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Now denote by  $S \in Isom(\mathbb{H}^2)$  the inversion with respect to the axis  $(x, y)$  (that is, the reflection whose axis is the one of  $\alpha_1$ ).

We now define  $h_n: \pi_1 \Sigma_g \rightarrow Isom^+(\mathbb{H}^3)$ , for every  $g \geq 3$ , as follows. Consider an isometric embedding  $i: \mathbb{H}^2 \hookrightarrow \mathbb{H}^3$ ; this determines an injection  $PSL(2, \mathbb{R}) \hookrightarrow Isom^+(\mathbb{H}^3)$ . Every reflection in  $\mathbb{H}^2$  can then be realized as a rotation in  $\mathbb{H}^3$ , and we denote again by  $\alpha_1, \beta_1, \alpha_2, \beta_2$  and  $S$  the corresponding elements of  $Isom^+(\mathbb{H}^3)$ . We put  $h_n(a_i) = \rho_n(a_i)$  and  $h_n(b_i) = \rho_n(b_i)$  for  $i = 1, 2$ , and we put  $h_n(a_i) = h_n(b_i) = S$  for  $3 \leq i \leq g$ .

We then have  $h_n \in R_g(3)$ , and  $h_n(b_1)$  is a hyperbolic element whose translation distance tends to  $+\infty$  as  $n \rightarrow +\infty$ , so that  $\lim_{n \rightarrow +\infty} d(h_n) = +\infty$ ; hence there exists an accumulation point  $(T, h_\infty) \in \overline{\partial m_g^u(3)}$  of this sequence of representations. We claim that no orientation on  $T$  is preserved by  $\rho_\infty$ . In order to prove this, it suffices to find a non degenerate tripod  $Trip(a, b, c, d) \subset T$ , with central point  $a$ , and an element  $\gamma \in \pi_1 \Sigma_g$ , such that  $\gamma(a) = a$ ,  $\gamma(b) = b$ ,  $\gamma(c) = d$  and  $\gamma(d) = c$ .

Note that if  $A, B \in Hyp$  do not have any common fixed points in  $\partial \mathbb{H}^2$ , then the attractive fixed point of  $AB^n$ , as  $n \rightarrow +\infty$ , converges to the one of  $B$ , whereas the repulsive fixed point of  $AB^n$  converges to the preimage, by  $A$ , of the one of  $B$ . Hence, the axis of  $\rho_n(b_1)$  converges to some fixed geodesic line in  $\mathbb{H}^2$ . Since the images  $\rho_n(\gamma)$  of the other generators  $\gamma \in \{a_1, a_2, b_2, \dots, a_g, b_g\}$  are fixed, there exists a sequence  $(x_0^n)_n \in (\mathbb{H}^2)^\mathbb{N}$  converging to a point  $x_\infty \in \mathbb{H}^2$ , such that for all  $n$ ,  $x_0^n \in \min(\rho_n)$  and  $i(x_0^n) \in \min(h_n)$ . Then  $d(i(x_0^n), h_n(a_3) \cdot i(x_0^n))$  is bounded. Moreover,  $h_n(b_1^{-1}) = \alpha_1^{-n} \beta_1^{-1}$ . Since the axis of the symmetry  $S = h_n(a_3)$  is the axis of the translation  $\alpha_1$ , we have

$$d(\alpha_1^{-n} \beta_1^{-1} i(x_0^n), S \cdot \alpha_1^{-n} \beta_1^{-1} i(x_0^n)) = d(\beta_1^{-1} i(x_0^n), S \beta_1^{-1} i(x_0^n)),$$

and this number is bounded, hence the distance  $d(h_n(b_1^{-1}) \cdot i(x_0^n), h_n(a_3 b_1^{-1}) \cdot i(x_0^n))$  is bounded. It follows from the construction of the  $\mathbb{R}$ -tree  $T$  given by M. Bestvina ([Bes88]) that there exist two distinct points of  $T$ ,  $x_0^\infty$  and  $h_\infty(b_1^{-1}) \cdot x_0^\infty$ , which are fixed by  $h_\infty(a_3)$ . Thus, the segment  $[x_0^\infty, \rho_\infty(b_1^{-1}) \cdot x_0^\infty]$  is globally fixed by  $h_\infty(a_3)$ . Since  $(h_\infty(a_3))^2 = id_T$ , in order to prove that there is a tripod  $Trip(a, b, c, d)$  such that  $h_\infty(a_3)(a) = a$ ,  $h_\infty(a_3)(b) = b$ ,  $h_\infty(a_3)(c) = d$  and  $h_\infty(a_3)(d) = c$ , it suffices to prove that  $h_\infty(a_3) \neq id_T$ . This follows for instance from the fact that  $h_n(a_2 b_1 a_2^{-1})$  is a hyperbolic element whose fixed points in  $\partial \mathbb{H}^2$  are distinct from those of  $\alpha_1$  and of  $h_n(b_1)$ , hence for  $n$  large enough the distance between  $h_n(a_2 b_1 a_2^{-1}) \cdot i(x_n)$  and the axis of the symmetry  $h_n(a_1)$  is of the order of  $d(h_n)$ : still by the construction of M. Bestvina, this yields a point of  $T$  which is not fixed by  $h_\infty(a_3)$ .  $\square$

### 3.3.2 The space $\overline{m_g^u}$ has at most 3 connected components

Now we shall exhibit another example of degeneracy. Fix an injective representation  $\rho: \pi_1 \Sigma_{g-1} \rightarrow PSL(2, \mathbb{R})$ . The circle  $\mathbb{S}^1$  being non numerable, there exists a point  $a_0 \in \mathbb{S}^1 = \partial \mathbb{H}^2$  such that for all  $\gamma \in \pi_1 \Sigma_{g-1}$ ,  $\rho(\gamma)a_0 = a_0 \Leftrightarrow \gamma = 1$ . Choose a point  $x_0 \in \mathbb{H}^2$ . Denote by  $A_n$  the hyperbolic element whose axis passes through  $x_0$ , with attractive point  $a_0$ , and translation length  $n$ . We define a representation  $\rho'_n: \pi_1 \Sigma_g \rightarrow PSL(2, \mathbb{R})$  by letting  $\rho'_n(a_i) = \rho(a_i)$ ,  $\rho'_n(b_i) = \rho(b_i)$  for  $i \leq g-1$ , and  $\rho'_n(a_g) = 1$ ,  $\rho'_n(b_g) = A_n$ .

**Proposition 3.32** *The sequence  $(\rho'_n)_n \in \overline{m_g^u}^\mathbb{N}$  converges to an action  $\rho_\infty$  on an  $\mathbb{R}$ -tree  $T$ , which does not depend on  $\rho$ .*

In [DK06], J. DeBlois and R. Kent proved that every connected components of  $R_{g-1}$  contain injective representations. It follows that this  $\mathbb{R}$ -tree is a common point to all the  $\partial m_{g,k}^u$  in  $\overline{m_g^u}$ , for every  $k \in \{0, \dots, 2g-4\}$ , since, of course, the representation  $\rho_n$  is also of Euler class  $k$  (this follows immediately from Milnor's algorithm, see formula 1), for all  $n \in \mathbb{N}$ . This proves the following result:

**Corollary 3.33** *Let  $g \geq 2$ . Then the space  $\overline{m_g^u(2)}$  has at most 3 connected components. More precisely, the components of Euler class between 0 and  $2g-4$  all meet at their boundary.*

Moreover, every injective representation in  $R_{g-1}$  of Euler class  $k$ , with  $|k| \leq 2g-5$ , is non elementary (indeed, elementary subgroups of  $PSL(2, \mathbb{R})$  are virtually abelian, hence they do not contain isomorphic copies of  $\pi_1 \Sigma_{g-1}$ ) and non discrete (by W. Goldman's corollary C [Gol88], faithful and discrete representations have Euler class  $2g-4$  or  $4-2g$ ). Hence, by theorem 2.39, every conjugacy class of injective representations gives rise to a distinct order. These conjugacy classes, by the theorem of DeBlois and Kent [DK06], have the same cardinality as  $\mathbb{R}$ . Hence,

**Corollary 3.34** *The surjective map  $\overline{m_g^o} \twoheadrightarrow \overline{m_g^u}$  has a fibre which has the cardinality of  $\mathbb{R}$ .*

It is known (see [Gol88]) that the space  $m_g^u(3)$  has two connected components, one containing all the representations of even Euler class in  $m_g^u(2)$ , and the other containing those of odd Euler class. The following result follows:

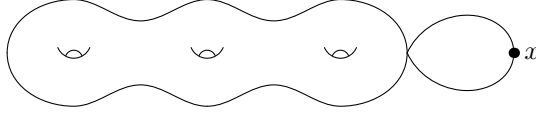
**Corollary 3.35** *Let  $g \geq 3$ . Then the space  $\overline{m_g^u(3)}$  is connected.*

**Proof** of proposition 3.32.

We shall first define an  $\mathbb{R}$ -tree  $T$  equipped with an action  $\rho_\infty$  of  $\pi_1 \Sigma_g$ ; we will then prove that  $(\mathbb{H}^2, \rho'_n) \rightarrow (T, \rho_\infty)$  in  $\overline{m_g^u}$ .

Let  $\mathcal{G}$  be the Cayley graph of the free product  $G = \pi_1 \Sigma_{g-1} * \mathbb{Z}$  for the generating set  $\{a_1, b_1, \dots, a_{g-1}, b_{g-1}, z\}$  where  $z$  is one of the two generators of the free factor  $\mathbb{Z}$  of  $G$ . Note that the action of  $G$  on  $\mathcal{G}$  is free, and that if  $e$  is the edge between 1 and  $z$ , then  $e$  separates  $\mathcal{G}$ , by definition of a free product. Let  $T$  be the dual graph, in  $\mathcal{G}$ , to the set  $\mathcal{E}$  of middles of edges of  $\mathcal{G}$ : its vertices are the connected components of  $\mathcal{G} \setminus \mathcal{E}$ , with an edge between two such connected components if their closures meet. We denote by  $\bar{1}, \bar{z}$  the vertices of  $T$  which are the images of the vertices 1 and  $z$  of  $\mathcal{G}$ , respectively. The action of  $G$  on  $\mathcal{G}$ , which is free and transitive on the vertices, induces an action of  $G$  on  $T$  which is transitive on the edges. The graph  $T$  is thus a tree, since each of its edges separates it, and the action of  $G$  on  $T$  is therefore minimal, with trivial edge stabilizers. Hence, it is a *geometric* action of the group  $G$  on the  $\mathbb{R}$ -tree  $T$ , in the sense of G. Levitt and F. Paulin (see [LP97]):  $G$  is the fundamental group of the bouquet of  $\Sigma_{g-1}$  with a circle; denote this

complex by  $S$ . Then the action of  $G$  on  $T$  is the action dual to the “lamination” defined by any point  $x$  of the circle of  $S$ .



The surjective morphism  $e: \pi_1 \Sigma_g \rightarrow G$  defined by  $e(a_i) = a_i$  and  $e(b_i) = b_i$  if  $i \leq g-1$ , and  $e(a_g) = 1$ ,  $e(b_g) = z$  induces a minimal action  $\rho_\infty$  of  $\pi_1 \Sigma_g$  on  $T$  by isometries, in which the stabilizer of edges is exactly the normal subgroup of  $\pi_1 \Sigma_g$  generated by  $a_g$ .

Now we shall prove that  $(\mathbb{H}^2, \rho'_n) \rightarrow (T, \rho_\infty)$ , in the sense of the equivariant Gromov topology. Let  $P$  be a finite subset of  $\pi_1 \Sigma_g$ . Let  $T_P$  be the minimal subtree of  $T$  containing the vertices  $\bar{1}, \bar{z} \in C \subset T$ , as well as  $P \cdot \bar{1}$  and  $P \cdot \bar{z}$ . Let  $\{x_1, \dots, x_m\}$  be a finite collection in  $T_P$  and let  $\varepsilon > 0$ . We need to prove that there exists  $\{x'_1, \dots, x'_m\} \subset \mathbb{H}^2$  such that for all  $i, j \in \{1, \dots, m\}$  and all  $\gamma_1, \gamma_2 \in P$ ,

$$|d(\rho'_n(\gamma_1) \cdot x'_i, \rho'_n(\gamma_2) \cdot x'_j) - d(\rho_\infty(\gamma_1) \cdot x_i, \rho_\infty(\gamma_2) \cdot x_j)| < \varepsilon.$$

For every  $\gamma \in P$ , the element  $e(\gamma) \in G$  can be uniquely written as  $e(\gamma) = u_1 z^{n_1} \dots u_k z^{n_k}$ , with  $u_1, \dots, u_k \in \pi_1 \Sigma_{g-1}$ ,  $u_2, \dots, u_k \neq 1$  and  $n_1, \dots, n_{k-1} \neq 0$ . Denote by  $Q \subset \pi_1 \Sigma_{g-1}$  the set described by all the elements  $u_1, \dots, u_k$  as  $\gamma$  describes  $P$ . Put

$$\alpha_P = \min_{\substack{u_1, u_2 \in Q \\ u_1 \neq u_2}} (\rho_0(u_1) a_0, \widehat{x_0}, \rho_0(u_2) a_0).$$

It is a non zero angle. Put also  $\beta_P = \max_{u \in Q} d_{\mathbb{H}^2}(x_0, \rho_0(u) \cdot x_0)$ .

If  $P$  is large enough, which we suppose now, the tree  $T_P$  is the orbit of the segment  $[\bar{1}, \bar{z}]$  under  $\rho_\infty(P)$ . For all  $i \in \{1, \dots, m\}$  there exists  $t_i \in [0, 1]$  and (at least one)  $c_i \in P$  such that  $x_i$  is on the segment  $[c_i \cdot \bar{1}, c_i \cdot \bar{z}]$  and such that  $d(x_i, c_i \cdot \bar{1}) = t_i = 1 - d(x_i, c_i \cdot \bar{z})$ . Denote by  $x'_i$  the element of  $\mathbb{H}^2$  such that  $\frac{1}{n} d_{\mathbb{H}^2}(x'_i, \rho'_n(c_i) x_0) = t = 1 - \frac{1}{n} d_{\mathbb{H}^2}(x'_i, \rho'_n(c_i) A_n x_0)$ . Such a point exists and is unique, since  $d_{\mathbb{H}^2}(\rho'_n(c_i) x_0, \rho'_n(c_i) A_n x_0) = n$ . By construction of the points  $x'_i$ , in order to check that for all  $i, j \in \{1, \dots, m\}$  and  $\gamma_1, \gamma_2 \in P$  we have

$$|d(\rho'_n(\gamma_1) \cdot x'_i, \rho'_n(\gamma_2) \cdot x'_j) - d(\rho_\infty(\gamma_1) \cdot x_i, \rho_\infty(\gamma_2) \cdot x_j)| < \varepsilon,$$

it suffices to do it in the case when  $x_i$  and  $x_j$  are vertices of the tree  $T$ . Suppose for instance that  $x_i = c_i \cdot \bar{1}$  and  $x_j = c_j \cdot \bar{1}$  with  $c_i, c_j \in P$ . Let  $\gamma_1, \gamma_2 \in P$ . Denote  $c = \gamma_1 c_1 (\gamma_2 c_2)^{-1}$  and  $e(c) = u_{k+1} t^{n_k} \dots u_2 t^{n_1} u_1$ , with  $n_1, \dots, n_k \neq 0$  and  $u_2, \dots, u_k \neq 1$ . Since the stabilizer of the edge  $\bar{1}$ , in the group  $G$ , is  $\pi_1 \Sigma_{g-1}$  and

since  $d(\bar{1}, \bar{z}) = 1$  in the tree  $T$ , we check that  $d(\rho_\infty(\gamma_1) x_i, \rho_\infty(\gamma_2) x_j) = \sum_{i=1}^k |n_i|$ .

We are going to prove that, asymptotically, the distance  $\frac{1}{n} d_{\mathbb{H}^2}(\rho'_n(c) x_0, x_0)$

approaches  $\sum_{i=1}^m |n_k|$ . We work by induction on  $k$ . Denote  $c' = u_k z^{n_{k-1}} \cdots u_2 z^{n_1}$ ,

and suppose that  $u_k \neq 1$  and that  $d_{\mathbb{H}^2}(\rho'_n(c)x_0, x_0) = n \sum_{i=1}^{k-1} |n_i| + O(1)$ . Then

$$\begin{aligned} d(\rho'_n(c')x_0, x_0) &= n \sum_{i=1}^{k-1} |n_i|, \text{ and hence } d(A_n^{n_k} \rho'_n(c')x_0, A_n^{n_k} x_0) = n \sum_{i=1}^{k-1} |n_i|, \text{ and} \\ d(A_n^{n_k} x_0, x_0) &= n |n_k|. \text{ Since } A_n^{n_k} \in PSL(2, \mathbb{R}) \text{ preserves the angles, we then} \\ \text{have } A_n^{n_k} \rho'_n(c')x_0, A_n^{n_k} x_0, x_0 &= \widehat{\rho'_n(c')x_0, x_0, A_n^{-n_k} x_0} \geq \frac{\alpha_{P'}}{2}, \text{ for } P' \text{ large enough} \\ \text{(but depending only on } P). \text{ This, combined with cosine law I, implies that} \\ d(A_n^{n_k} \rho'_n(c')x_0, x_0) &= n \sum_{i=1}^m |k_i| + O(1). \text{ Since } d(u_{k+1} x_0, x_0) \leq \beta, \text{ it follows that} \\ d(\rho'_n(c)x_0, x_0) &= n \sum_{i=1}^k |n_i| + O(1), \text{ which completes the proof. } \quad \square \end{aligned}$$

### 3.3.3 The space $\overline{m}_g^u$ is connected

We will now prove theorem 1.2.

#### Theorem 3.36

- For all  $g \geq 2$ , the space  $\overline{m}_g^u$  has at most two connected components. More precisely, all the connected components, except possibly the one of Euler class  $2g - 3$ , meet at their boundaries.
- For all  $g \geq 4$ , the space  $\overline{m}_g^u$  is connected.

We still consider the generating set  $S = \{a_1, \dots, b_g\}$  of  $\pi_1 \Sigma_g$ . The main idea is the following.

**Lemma 3.37** *Let  $\Gamma$  be a finitely generated group and let  $\phi: \Gamma \rightarrow PSL(2, \mathbb{R})$  be a discrete, faithful representation of cocompact image. Denote by  $B_n$  the closed ball of radius  $n$  for the Cayley metric on  $\pi_1 \Sigma_g$  for the generating set  $S$ , and suppose that  $\phi_n: \pi_1 \Sigma_g \rightarrow \Gamma$  is a non injective morphism such that  $\text{Ker}(\phi_n) \cap B_n = \{1\}$ . Then  $d(\phi \circ \phi_n) \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Let  $(T, \rho_\infty)$  be an accumulation point in  $\partial \overline{m}_g^u$  of the sequence  $(\phi \circ \phi_n)_{n \in \mathbb{N}}$ . Then the action  $\rho_\infty$  has small edge stabilizers.*

#### Proof

First, let us prove that  $\lim_{n \rightarrow +\infty} d(\phi \circ \phi_n) = +\infty$ . By contradiction, suppose that we can extract a subsequence such that  $(d(\phi \circ \phi_{\varphi(n)}))_{n \in \mathbb{N}}$  converges to a real number  $d \in \mathbb{R}_+$ . Fix a point  $x_0 \in \mathbb{H}^2$ . Since the image  $\phi(\Gamma)$  is cocompact, there exist  $g_{\varphi(n)} \in \phi(\Gamma)$  and

$$x_n \in \min \left( g_{\varphi(n)} \cdot (\phi \circ \phi_{\varphi(n)}) \cdot g_{\varphi(n)}^{-1} \right)$$

such that the distance  $d(x_0, x_n)$  is unbounded; say  $d(x_0, x_n) \leq k$ . Denote by  $\rho_n$  the representation  $g_{\varphi(n)} \cdot (\phi \circ \phi_{\varphi(n)}) \cdot g_{\varphi(n)}^{-1}$ . Then for every  $n \geq 0$ ,  $\rho_n$  is discrete, and  $\text{Ker}(\rho_n) \cap B_{\varphi(n)} = \{1\}$ . For every  $n \in \mathbb{N}$  and every  $\gamma \in S$ ,  $d(\rho_n(\gamma) \cdot x_0, x_0) \leq d(\rho_n) + 2k$ , and  $\lim_n d(\rho_n) = d$  hence, by proposition 2.2, up to extract it,  $(\rho_n)_n$  converges to a representation  $\rho \in R_g$ . Since the set  $\rho_n(\pi_1 \Sigma_g) = \phi(\Gamma) \subset PSL(2, \mathbb{R})$  is discrete and since for all  $\gamma \in \pi_1 \Sigma_g \setminus \{1\}$ , for  $n$  large enough  $\rho_n(\gamma) \neq \text{Id}$ , we have  $\rho(\gamma) \neq \text{Id}$ , hence  $\rho$  is injective. Hence, the representation  $\rho$  is discrete and faithful, hence  $|e(\rho)| = 2g - 2$ , thus  $\rho$  is the limit of representations  $\rho_n$  which are non injective, in particular  $|e(\rho_n)| \neq 2g - 2$ : this is in contradiction with the continuity of the Euler class (in fact it is not necessary to use the Euler class here, but it gives the shortest proof).

Now denote  $\rho_n = \phi \circ \phi_n$ . It follows that there exists an accumulation point  $(T, \rho_\infty) \in \partial \overline{m_g^u}$  of the sequence  $(\rho_n)_{n \in \mathbb{N}}$ . We still have to prove that this action has small edge stabilizers. But our representation  $\rho_n$ , for all  $n \geq 0$ , is *discrete*. We can therefore apply the same argument as M. Bestvina and F. Paulin ([Bes88, Pau88]), consisting of applying Margulis' lemma. Here we follow the lines of the proof of theorem 6.7 of [Pau88], pages 78-79, and refer the reader to this text for more details.

**Margulis' lemma:** *There exists a constant  $\mu > 0$ , depending only on  $n$ , such that for every discrete group  $\Gamma$  of isometries of  $\mathbb{H}^n$ , and for all  $x \in \mathbb{H}^n$ , the subgroup generated by  $\{\gamma \in \Gamma \mid d(x, \gamma x) < \mu\}$  is virtually abelian.*

Let us suppose that there exists a segment  $[x_\infty, y_\infty]$  of the limit  $\mathbb{R}$ -tree, whose stabilizer contains a free group of rank 2. Up to considering a subgroup of index 2, we may suppose that there exists a free group of rank 2,  $\langle \alpha, \beta \rangle \subset \pi_1 \Sigma_g$ , such that  $\alpha$  and  $\beta$  fix  $x_\infty$  and  $y_\infty$ . Hence, for all  $\varepsilon > 0$ , and for  $n$  large enough, there exist  $x_n, y_n \in \mathbb{H}^2$  such that  $\left| \frac{1}{\ell(\rho_n)} d_{\mathbb{H}^2}(x_n, y_n) - 1 \right| < \varepsilon$ ,  $d_{\mathbb{H}^2}(x_n, \rho_n(\alpha)x_n) < \varepsilon \ell(\rho_n)$ ,  $d_{\mathbb{H}^2}(y_n, \rho_n(\alpha)y_n) < \varepsilon \ell(\rho_n)$ , and similarly for  $\rho_n(\beta)$ . Denote by  $z_n$  the middle of the segment  $[x_n, y_n]$ : it can be proved (see [Pau88]) that if  $\varepsilon$  is small enough and if  $\ell(\rho_n)$  is large enough, then the elements  $[\rho_n(\alpha), \rho_n(\beta)]$  and  $[\rho_n(\alpha^2), \rho_n(\beta)]$  move the point  $z_n$  by a distance less than  $\mu$ . By Margulis' lemma, the elements  $\rho_n([\alpha, \beta]), \rho_n([\alpha^2, \beta]) \in PSL(2, \mathbb{R})$  generate a virtually abelian subgroup of  $PSL(2, \mathbb{R})$ . But it is an easy exercise (see e.g. [Wol07], lemme 1.1.18) to check that virtually abelian subgroups of  $PSL(2, \mathbb{R})$  are *metabelian*, i.e., all the commutators commute. In particular, denote for instance  $w(\alpha, \beta) = [[[\alpha, \beta], [\alpha^2, \beta]], [[\alpha, \beta]^2, [\alpha^2, \beta]]]$ : then we have  $\rho_n(w(\alpha, \beta)) = 1$  for all  $n$  large enough. If  $n$  is large enough, and larger than the length of the word  $w(\alpha, \beta)$  in the generators  $a_i, b_i$ , this implies that  $w(\alpha, \beta) = 1$  in  $\pi_1 \Sigma_g$  (since  $\text{Ker}(\rho_n) \cap B_n = \{1\}$ ), which contradicts the fact that  $\langle \alpha, \beta \rangle$  is free. Hence, the action  $\rho_\infty$  indeed has small edge stabilizers.  $\square$

**Proof of theorem 3.36.**

- Fix a cocompact Fuchsian group  $\Gamma$ , a subgroup  $\mathbb{F}_2 \subset \Gamma$  isomorphic to a free group of rank 2, and  $\phi: \Gamma \rightarrow PSL(2, \mathbb{R})$  the tautologic representation (the inclusion). Consider the morphism  $\phi_n: \pi_1 \Sigma_g \rightarrow \mathbb{F}_2$  given by theorem 2.42, and let  $\rho_n = \phi \circ \phi_n$ . Then  $\rho_n$  factors through the free

group, which has a trivial  $H^2$ , hence by remark 2.34, the representation  $\rho_n$  has Euler class zero. By lemma 3.37,  $(\rho_n)_{n \in \mathbb{N}}$  possesses an accumulation point  $(T, \rho_\infty) \in \partial \overline{m}_g^u$ , which has small edge stabilizers. By R. Skora's theorem [Sko96], this implies that this limit is also at the boundary of the Teichmüller space; in other words, this action on an  $\mathbb{R}$ -tree is also the limit of representations of Euler class  $2g-2$ . Hence, the closures of the connected components of  $m_g^u$  of Euler classes 0 and  $2g-2$  meet. By corollary 3.33, we already knew that the connected components of Euler classes 0, 1,  $\dots$ ,  $2g-4$  meet at their boundaries; this concludes the proof of the first point.

- By proposition 4.5 of [FW07], the map  $p_g: \pi_1 \Sigma_g \rightarrow PSL(2, \mathbb{R})$  that we defined in section 2.5 is discrete and has Euler class  $2g-3$ . We have seen (lemma 2.43) that for all  $n \geq 0$ , there exists  $\phi_n \in \text{Aut}(\pi_1 \Sigma_g)$  such that  $\text{Ker}(p_g \circ \phi_n) \cap B_n = \{1\}$ . The representation  $p_g \circ \phi_n$  is still discrete, and we have  $|e(p_g \circ \phi_n)| = 2g-3$ , hence, as before, the lemma 3.37 ensures that the connected components of  $m_g^u$  of Euler classes  $2g-3$  and  $2g-2$  meet at their boundaries.

□

**Remark 3.38** For all  $g \geq 2$ , consider the action  $(\rho_\infty, T)$  exhibited in proposition 3.32. The representation  $\rho_\infty: \pi_1 \Sigma_g \rightarrow \text{Isom}(T)$  factors through the group  $\pi_1 \Sigma_{g-1} * \mathbb{Z}$ , which acts on  $T$  with trivial arc stabilizers. Since there exists a morphism  $\pi_1 \Sigma_{g-1} * \mathbb{Z} \rightarrow \mathbb{F}_2$  with non abelian image to some free group of rank 2, as in section 2.5 we can prove that there exist automorphisms  $\phi_n$  of  $\pi_1 \Sigma_g$  such that  $\text{Ker}(\rho_\infty \circ \phi_n) \cap B_n = \{1\}$ . Following the proof of lemma 5.7 of [Pau88] (see also remark (1), page 73 in [Pau88]), we can prove that up to extract it, the sequence  $(\rho_\infty \circ \phi_n, T)$  converges to an action on an  $\mathbb{R}$ -tree with small stabilizers. This proves that for all  $g \geq 2$  and all  $k \in \{0, \dots, 2g-4\}$ ,  $\overline{m}_{g,k}^u \cap \overline{m}_{g,2g-2}^u \neq \emptyset$ .

Hence, for all  $g \geq 4$  and all  $k \in \{0, \dots, 2g-3\}$ , we have  $\overline{m}_{g,k}^u \cap \overline{m}_{g,2g-2}^u \neq \emptyset$ .

### 3.3.4 Dynamics

Finally, here we complete the proof of theorem 1.1, which implies that the compactification  $\overline{m}_g^u$  is *extremely* degenerated, pledging that  $\overline{m}_g^o$  is a more natural object to consider and to try to generalize to higher dimensions and to the compactification of other representation spaces.

**Proposition 3.39** Let  $g \geq 4$  and  $k \in \{0, \dots, 2g-3\}$ . Then the boundary of the Teichmüller space embeds in  $\partial m_{g,k}^u \subset \overline{m}_g^u$  as a closed, nowhere dense subset.

#### Proof

Fix  $g \geq 4$  and  $k \in \{0, \dots, 2g-3\}$ . Put  $F_k = \partial m_{g,2g-2}^u \cap \partial m_{g,k}^u$ . By remark 3.38, we have  $F_k \neq \emptyset$ . Since  $m_{g,2g-2}^u$  and  $m_{g,k}^u$  are invariant under the (natural)

action of  $Out(\pi_1 \Sigma_g)$ , it follows that  $F_k$  is invariant, too, under this action. It is well-known (see [FLP91], exposé 6, theorem VII.2, p. 117 ; see also [Mas85]) that the action of  $Out(\pi_1 \Sigma_g)$  on  $\partial m_{g,2g-2}^u$  is minimal, that is, every closed subset of  $\partial m_{g,2g-2}^u$ , invariant under  $Out(\pi_1 \Sigma_g)$ , is either empty or is  $m_{g,2g-2}^u$  itself. Since  $F_k$  is a closed subset, this implies that  $\partial m_{g,2g-2}^u = F_k \subset \partial m_{g,k}^u$ .

Now denote by  $G_k$  the boundary of  $\partial m_{g,2g-2}^u$  in the space  $\partial m_{g,k}^u$ . We can easily produce elements in  $\partial m_{g,k}^u$  which do not have small stabilizers (if  $k \leq 2g-4$  then the tree  $(\rho_\infty, T_\infty)$  of proposition 3.32 is an example; if  $k = 2g-3$  then we can compose the map  $p_g$  with Dehn twists along the last handle: this does not touch the kernel of the map  $p_g$ , hence this yields a sequence of actions with a fixed non trivial kernel in  $\pi_1 \Sigma_g$ , converging (up to extract it) to an action on an  $\mathbb{R}$ -tree, with this non trivial kernel). Hence,  $\partial m_{g,k}^u \neq \partial m_{g,2g-2}^u$ . By proposition 3.2, the space  $m_{g,k}^u$  is connected: it follows that  $G_k \neq \emptyset$ . Since  $\partial m_{g,2g-2}^u$  is closed, we have  $G_k \subset \partial m_{g,2g-2}^u$ , and  $G_k$  is again invariant under the action of the mapping class group. Hence,  $G_k = \partial m_{g,2g-2}^u$ .  $\square$

## References

- [AM85] Roger C. Alperin and Kenneth N. Moss. Complete trees for groups with a real-valued length function. *J. London Math. Soc. (2)*, 31(1):55–68, 1985.
- [Bau62] Gilbert Baumslag. On generalised free products. *Math. Z.*, 78:423–438, 1962.
- [Bes88] Mladen Bestvina. Degenerations of the hyperbolic space. *Duke Math. J.*, 56(1):143–161, 1988.
- [Bou71] N. Bourbaki. *Topologie générale, chapitres 1 à 4*. Hermann, Paris, 1971.
- [BP92] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992.
- [Cal04] Danny Calegari. Circular groups, planar groups, and the Euler class. In *Proceedings of the Casson Fest*, volume 7 of *Geom. Topol. Monogr.*, pages 431–491 (electronic). Geom. Topol. Publ., Coventry, 2004.
- [CDP89] M. Coornaert, T. Delzant, and A. Papadopoulos. *Notes sur les groupes hyperboliques de Gromov*. I.R.M.A., Strasbourg, 1989.
- [CG05] Christophe Champetier and Vincent Guirardel. Limit groups as limits of free groups. *Israel J. Math.*, 146:1–75, 2005.
- [CS83] Marc Culler and Peter B. Shalen. Varieties of group representations and splittings of 3-manifolds. *Ann. of Math. (2)*, 117(1):109–146, 1983.



- [DK06] Jason DeBlois and Richard P. Kent, IV. Surface groups are frequently faithful. *Duke Math. J.*, 131(2):351–362, 2006.
- [FLP91] Fathi, Laudenbach, and Poenaru, editors. *Travaux de Thurston sur les surfaces*. Société Mathématique de France, Paris, 1991. Séminaire Orsay, Reprint of *Travaux de Thurston sur les surfaces*, Soc. Math. France, Paris, 1979 [MR0568308 (82m:57003)], Astérisque No. 66-67 (1991).
- [Fre96] Jean Fresnel. *Méthodes modernes en géométrie*. Hermann, Paris, 1996.
- [FW07] Louis Funar and Maxime Wolff. Non-injective representations of a closed surface group into  $\mathrm{PSL}(2, \mathbb{R})$ . *Math. Proc. Cambridge Philos. Soc.*, 142(2):289–304, 2007.
- [Ghy87a] Étienne Ghys. Classe d’Euler et minimal exceptionnel. *Topology*, 26(1):93–105, 1987.
- [Ghy87b] Étienne Ghys. Groupes d’homéomorphismes du cercle et cohomologie bornée. In *The Lefschetz centennial conference, Part III (Mexico City, 1984)*, volume 58 of *Contemp. Math.*, pages 81–106. Amer. Math. Soc., Providence, RI, 1987.
- [Gol80] William M. Goldman. *Discontinuous groups and the Euler class*. PhD thesis, Berkeley, 1980.
- [Gol84] William M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. in Math.*, 54(2):200–225, 1984.
- [Gol88] William M. Goldman. Topological components of spaces of representations. *Invent. Math.*, 93(3):557–607, 1988.
- [Gui98] Vincent Guirardel. Approximations of stable actions on  $\mathbf{R}$ -trees. *Comment. Math. Helv.*, 73(1):89–121, 1998.
- [Gui04] Vincent Guirardel. Limit groups and groups acting freely on  $\mathbb{R}^n$ -trees. *Geom. Topol.*, 8:1427–1470 (electronic), 2004.
- [Hit87] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.
- [Kat92] Svetlana Katok. *Fuchsian groups*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.
- [KL95] M. Kapovich and B. Leeb. On asymptotic cones and quasi-isometry classes of fundamental groups of 3-manifolds. *Geom. Funct. Anal.*, 5(3):582–603, 1995.
- [LP97] Gilbert Levitt and Frédéric Paulin. Geometric group actions on trees. *Amer. J. Math.*, 119(1):83–102, 1997.

- [Mas85] Howard Masur. Ergodic actions of the mapping class group. *Proc. Amer. Math. Soc.*, 94(3):455–459, 1985.
- [Mat87] Shigenori Matsumoto. Some remarks on foliated  $S^1$  bundles. *Invent. Math.*, 90(2):343–358, 1987.
- [McM] C. McMullen. Ribbon  $\mathbb{R}$ -trees and holomorphic dynamics on the unit disk. Preprint.
- [Mil58] John Milnor. On the existence of a connection with curvature zero. *Comment. Math. Helv.*, 32:215–223, 1958.
- [ML95] Saunders Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.
- [Mor86] John W. Morgan. Group actions on trees and the compactification of the space of classes of  $\mathrm{SO}(n, 1)$ -representations. *Topology*, 25(1):1–33, 1986.
- [MS84] John W. Morgan and Peter B. Shalen. Valuations, trees, and degenerations of hyperbolic structures. I. *Ann. of Math. (2)*, 120(3):401–476, 1984.
- [Par] Anne Parreau. Compactification d’espaces de représentations de groupes de type fini. Preprint, available on <http://www-fourier.ujf-grenoble.fr/~parreau>.
- [Pau88] Frédéric Paulin. Topologie de Gromov équivariante, structures hyperboliques et arbres réels. *Invent. Math.*, 94(1):53–80, 1988.
- [Pau89] Frédéric Paulin. The Gromov topology on  $\mathbf{R}$ -trees. *Topology Appl.*, 32(3):197–221, 1989.
- [Pau04] Frédéric Paulin. Sur la compactification de Thurston de l’espace de Teichmüller. To appear in “Géométries à courbure négative ou nulle, groupes discrets et rigidités (Actes de l’école d’été de l’Institut Fourier, Grenoble)”. *Séminaires et Congrès*, (18), 2004.
- [Sel01] Zlil Sela. Diophantine geometry over groups. I. Makanin-Razborov diagrams. *Publ. Math. Inst. Hautes Études Sci.*, 93:31–105, 2001.
- [Sko96] Richard K. Skora. Splittings of surfaces. *J. Amer. Math. Soc.*, 9(2):605–616, 1996.
- [Thu] W.P. Thurston. 3-manifolds, foliations and circles II. Preprint.
- [Thu88] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.

- [Wol07] Maxime Wolff. *Sur les composantes exotiques des espaces d'actions de groupes de surfaces sur le plan hyperbolique*. PhD thesis, Université de Grenoble I, 2007.
- [Woo71] John W. Wood. Bundles with totally disconnected structure group. *Comment. Math. Helv.*, 46:257–273, 1971.